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ON SIEGEL DISKS OF A CLASS OF ENTIRE MAPS

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To the memory of Adrien Douady (1935-2006)

ABSTRACT. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire map of the form $f(z) = P(z) \exp(Q(z))$, where P and Q are polynomials of arbitrary degrees (we allow the case $Q = 0$). Building upon a method pioneered by M. Shishikura, we show that if f has a Siegel disk of bounded type rotation number centered at the origin, then the boundary of this Siegel disk is a quasicircle containing at least one critical point of f. This unifies and generalizes several previously known results.

CONTENTS

1. INTRODUCTION

Let f be a non-linear entire map of the complex plane or a rational map of the Riemann sphere of degree ≥ 2. Suppose $f(0) = 0$ and $f'(0) = e^{2\pi i\theta}$, where the *rotation number* $0 < \theta < 1$ is irrational. We say f is *locally linearizable* at the fixed point 0 if there exists a holomorphic change of coordinates near 0 which conjugates f to its linear part $R_\theta : z \mapsto e^{2\pi i\theta}z$. The maximal region in which f is conjugate to R_θ is a simply-connected domain Δ_f called the *Siegel disk* of f centered at 0. Thus f acts as an irrational rotation in Δ_f. However, understanding the topology and geometry of the boundary $\partial\Delta_f$, and the dynamics of f on it, is often quite difficult.

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This paper will study Siegel disks in the family $\mathcal{E}^{p,q}$ of entire maps of the form

$$f : z \mapsto P(z) \exp(Q(z)), \tag{1.1}$$

where P and Q are polynomials of degrees p and q, respectively. We consider the subfamily $\mathcal{E}^{p,q}(\theta) \subset \mathcal{E}^{p,q}$ of maps which have a Siegel disk of rotation number θ centered at the origin. There are good reasons to view these entire maps as close relatives of polynomials. For example, they have finitely many zeros and critical points and, in the transcendental case $q > 0$, a single (finite) asymptotic value at the origin. They belong to the Speiser class $\mathcal{S}$ of entire maps with finitely many singular values, or more generally to the Eremenko-Lyubich class $\mathcal{B}$ of entire maps with a bounded set of singular values, which are known to share many of the dynamical properties of polynomial maps (see [**EL**] and compare [**MNTU**] where such maps are called "decorated exponential"). Our primary focus will of course be on the transcendental case $q > 0$, but the analysis will cover the polynomial case $q = 0$ as well.

Expand the rotation number θ into its continued fraction $[a_1, a_2, a_3, \ldots]$, where each a_n is a positive integer. Recall that θ is of ***bounded type*** if $\{a_n\}$ is a bounded sequence. It is well-known that in this case f is locally linearizable at the origin.

Main Theorem. *Let $f \in \mathcal{E}^{p,q}(\theta)$, where $0 < \theta < 1$ is an irrational number of bounded type. Then the Siegel disk of f centered at the origin is bounded by a quasicircle in the plane which contains at least one critical point of f.*

Compare Fig. 1.

This generalizes and unifies several results obtained over the past 20 years by various authors. They include Douady-Herman-Swiatek's for quadratic polynomials [**D**], this author's for cubic polynomials [**Z1**], Shishikura's for polynomials of arbitrary degree (unpublished), Geyer's for the map $z \mapsto e^{2\pi i\theta} z\, e^z$ [**Ge**], and Keen-Zhang's for the maps of the form $z \mapsto (e^{2\pi i\theta} z + az^2)\, e^z$ [**KZ**]. See also Chéritat's examples of "simple" entire maps in [**C**].

It is important to realize that the choice of normalization for the family $\mathcal{E}^{p,q}(\theta)$ in the transcendental case $q > 0$ is not a matter of convenience. In fact, in contrast to the polynomial case, when $q > 0$ the space $\mathcal{E}^{p,q}$ is not invariant under affine conjugations that move the origin. As a result, the Main Theorem does not imply anything about bounded type Siegel disks in $\mathcal{E}^{p,q}$ that are centered at points other than 0. This is not entirely a shortcoming of our approach: for example, if θ is an irrational of bounded type and $\lambda = e^{2\pi i\theta}$, the boundary of the Siegel disk centered at λ of the map $z \mapsto \lambda\, e^{z-\lambda}$ in $\mathcal{E}^{0,1}$ contains ∞, hence fails to be a Jordan curve on the sphere (see [**H2**] and compare Fig. 2). This phenomenon is known to be common to all entire maps without critical points [**GS**].

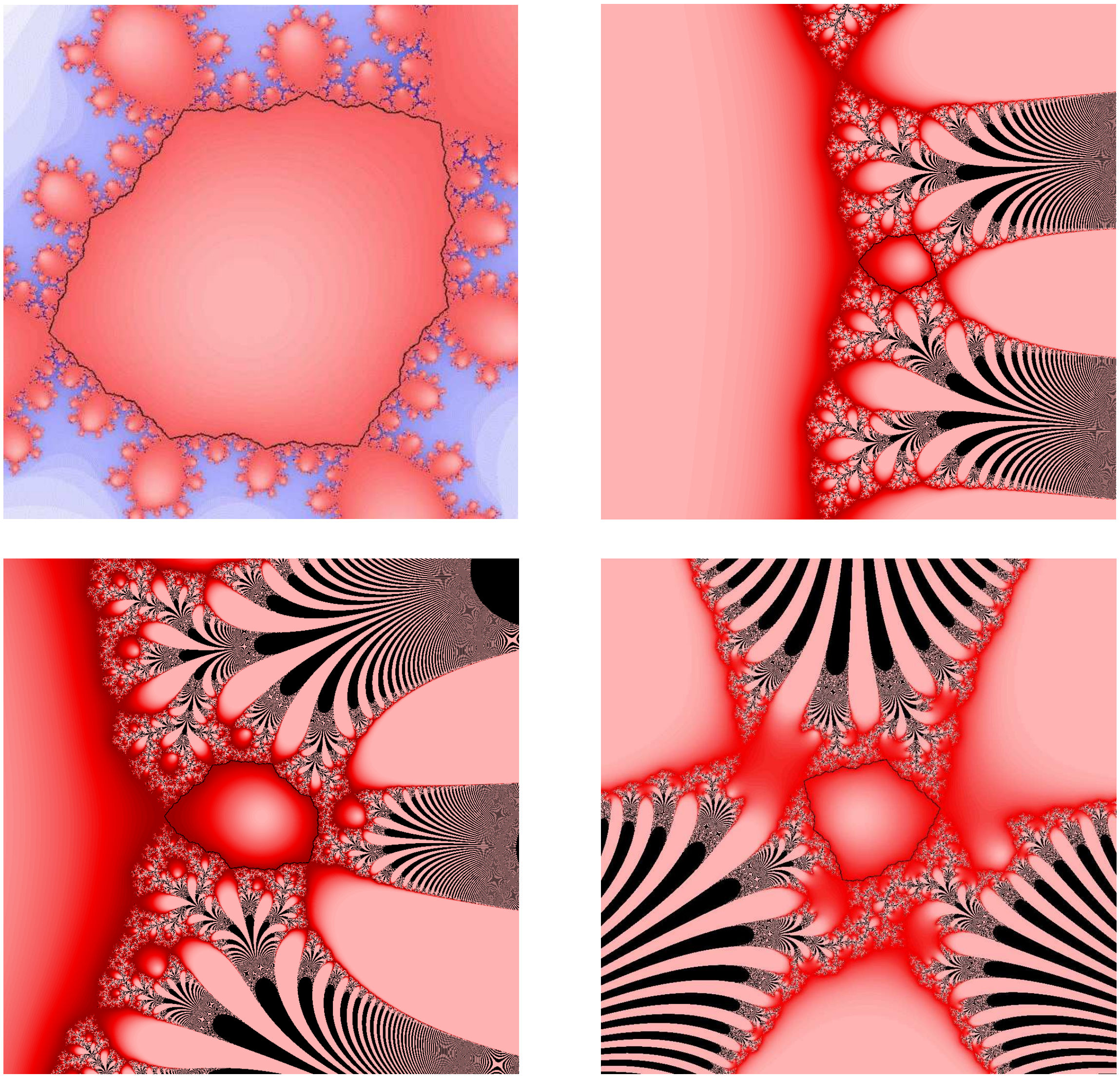

FIGURE 1. *The Julia sets of four maps in* $\mathcal{E}^{p,q}(\theta)$. *In each case, the boundary of the Siegel disk is the quasicircle delineated in black at the center of picture. Upper left:* $z \mapsto \lambda z + z^2$; *upper right:* $z \mapsto \lambda z\, e^z$; *lower left:* $z \mapsto \lambda z\,(1 - 2z/3)\, e^z$; *lower right:* $z \mapsto \lambda z\,(1 - (11+3i)z/13)\, e^{iz^3}$. *Here* $\lambda = e^{\pi i(\sqrt{5}-1)}$ *corresponds to the golden mean rotation number.*

Our strategy of proof is strongly inspired by Shishikura's unpublished work for Siegel disks of polynomials. Let $f \in \mathcal{E}^{p,q}(\theta)$ and $\zeta_f : \mathbb{D} \to \Delta_f$ be the unique conformal isomorphism that satisfies $\zeta_f(0) = 0, \zeta_f'(0) > 0$. It follows from Schwarz Lemma that

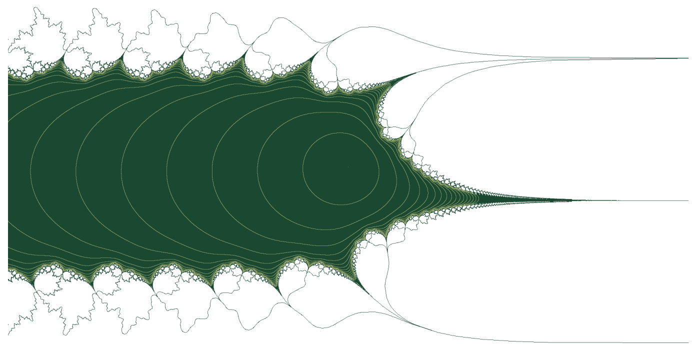

FIGURE 2. *(Courtesy of A. Chéritat) Invariant curves in the Siegel disk of the map* $z \mapsto \lambda\, e^{z-\lambda}$ *centered at* λ*. As before,* $\lambda = e^{\pi i(\sqrt{5}-1)}$*. This Siegel disk is unbounded and its boundary fails to be locally connected.*

ζ_f linearizes f in the sense that

$$f \circ \zeta_f = \zeta_f \circ R_\theta \qquad \text{in } \mathbb{D}.$$

Following Shishikura, we show that the invariant curves $\gamma_{f,r} := \zeta_f(\{z : |z| = r\})$ in Δ_f are K-quasicircles for a constant $K > 1$ independent of the radius $0 < r < 1$. A simple compactness argument then proves that $\partial\Delta_f$ is a quasicircle (Theorem 2.3). That $\partial\Delta_f$ must contain a critical point follows from a standard argument (Theorem 2.8).

Let us give a quick outline of the proof: Fix $0 < r < 1$ and take a suitable quasiconformal reflection $I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which swaps 0 and ∞ and keeps the invariant curve $\gamma_{f,r} \subset \Delta_f$ fixed pointwise. Use I to "symmetrize" f about $\gamma_{f,r}$ in order to produce a quasiregular dynamics $F : \mathbb{C}^* \to \widehat{\mathbb{C}}$ which commutes with I. This replaces the Siegel disk of f with a "quasiconformal Herman ring" for F. The sphere admits a conformal structure μ of bounded dilatation which is invariant under both F and I. Straightening μ by an appropriately normalized quasiconformal map $\xi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ gives a conjugate map $G := \xi \circ F \circ \xi^{-1} : \mathbb{C}^* \to \widehat{\mathbb{C}}$ which is holomorphic and commutes with the reflection $z \mapsto 1/\overline{z}$ across the unit circle $\mathbb{T} = \xi(\gamma_{f,r})$. The map G has a genuine Herman ring which contains $\mathbb{T}$ as an invariant curve. Note, however, that the maximal dilatation of ξ may depend on r and *a priori* can grow large as $r \to 1$.

By analyzing the explicit form of G and estimating the location of its poles, we show that there are constants $\delta > 1$ and $M > 0$, *depending only on the degrees* p *and* q, such that $|zG'(z)/G(z)| \leq M$ in the annulus $\{z : \delta^{-1} < |z| < \delta\}$ (Theorem 5.6).

This step is rather easy for polynomials but requires some work in the transcendental case. Since the rotation number θ is assumed to be of bounded type, the theorem of Herman-Swiatek (Theorem 2.7) shows that the restriction of G to $\mathbb{T}$ is k-quasisymmetrically conjugate to R_θ for a constant $k > 1$ which only depends on p, q, θ. Extend this conjugacy to a K-quasiconformal map $\mathbb{D} \to \mathbb{D}$, with $K > 1$ independent of r, and use it to replace the action of G on $\mathbb{D}$ with a K-quasiconformal rotation by angle θ. Intuitively, we paste a "quasiconformal Siegel disk" on $\mathbb{D}$ to produce a new quasiregular dynamics $\hat{G} : \mathbb{C} \to \mathbb{C}$. The map $\hat{G}$ admits an invariant conformal structure ν of bounded dilatation. Straightening ν by an appropriately normalized K-quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ gives an entire map $g := \psi \circ \hat{G} \circ \psi^{-1} : \mathbb{C} \to \mathbb{C}$. It is easily verified that $g \in \mathcal{E}^{p,q}(\theta)$ and that the K-quasicircle $\psi(\mathbb{T})$ is just the invariant curve $\gamma_{g,r} = \zeta_g(\{z : |z| = r\})$ in the Siegel disk Δ_g. If we could show that g is the same map f that we started with, it would follow that $\gamma_{f,r}$ is a K-quasicircle for a constant K independent of r, which would prove the Main Theorem.

Unfortunately, the rigidity property $g = f$ is too good to be true for a general $f \in \mathcal{E}^{p,q}(\theta)$. The procedure

$$f \quad \overset{\text{symmetrize}}{\longmapsto} \quad F \quad \overset{\text{straighten}}{\longmapsto} \quad G \quad \overset{\text{modify}}{\longmapsto} \quad \hat{G} \quad \overset{\text{straighten}}{\longmapsto} \quad g$$

defines a ***surgery map*** $\mathcal{S}_r : \mathcal{E}^{p,q}(\theta) \to \mathcal{E}^{p,q}(\theta)$ which is far from the identity. In fact, the map $g = \mathcal{S}_r(f)$ may not be even topologically conjugate to f, and even when it is, one may not be able to promote the conjugacy to a conformal one. The difficulty arises when f has a critical point that is ***captured*** by its Siegel disk, in the sense that its forward orbit eventually hits Δ_f. Let us call the first point of hitting a ***capture spot*** of f in Δ_f. Then, a necessary and sufficient condition for the existence of a conformal conjugacy between f and $g = \mathcal{S}_r(f)$ is that the capture spots of f and g have the same conformal positions in their respective Siegel disks Δ_f and Δ_g (Theorem 6.1). This can hardly be guaranteed in the above surgery.

To circumvent this problem, we separate the argument into three cases based on the number and position of the capture spots of f in Δ_f:

• *Case 1.* The only capture spot of f, if any at all, is the fixed point 0. In other words, every critical orbit is either disjoint from Δ_f or lands at the origin. This case is easy to handle since a standard pull-back argument shows that f is rigid, so $\mathcal{S}_r(f) = f$ (Corollary 6.2).

• *Case 2.* There is precisely one non-zero capture spot of f in Δ_f. In other words, there is an $\omega \in \Delta_f \smallsetminus \{0\}$ such that the forward orbit of every captured critical point hits Δ_f for the first time at ω or else at 0. In this case, we produce a holomorphic family $\{f_t\}_{t \in \mathbb{D}^*}$ of quasiconformal deformations of f in $\mathcal{E}^{p,q}(\theta)$ with the property that the non-zero capture spot ω_t of f_t has conformal position t in Δ_{f_t} (Theorem 7.1). We use holomorphic motions to show that there is a constant K depending only on

p, q, θ such that the invariant curve $\gamma_{t,r} := \zeta_{f_t}(\{z : |z| = r\}) \subset \Delta_{f_t}$ is a K-quasicircle whenever $0 < |t| < 1/2$ or $r < |t| < 1$ (Lemma 7.4 and Lemma 7.6). The case of the intermediate values of $|t|$ is then handled by applying the Maximum Modulus Principle to a suitable cross-ratio function $\mathbb{D}^* \to \mathbb{C}$ (Theorem 7.7).

• *Case 3.* For the general case, let U be an iterated preimage of Δ_f which contains m critical points counting multiplicities. We modify the dynamics of f on an appropriate subset of U so that the new map $U \to f(U)$ is a quasiregular branched covering with a single branch point of order m. We apply this type of modification to all such U, making sure that the resulting branch points eventually map to 0 or some designated point $\omega \in \Delta_f \smallsetminus \{0\}$. Straightening the resulting quasiregular action, we obtain a map $g \in \mathcal{E}^{p,q}(\theta)$ which falls into one of the categories covered by the cases (1) or (2) above. The maps f and g are not topologically conjugate, but there is a quasiconformal homeomorphism of the plane which maps $\partial\Delta_f$ to $\partial\Delta_g$. Since $\partial\Delta_g$ is a quasicircle by the cases (1) or (2), it follows that $\partial\Delta_f$ is a quasicircle as well, which completes the proof of the Main Theorem.

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2. Preliminaries

Throughout the paper we will adopt the following notations:

- $\mathbb{C}$ is the complex plane and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere.
- $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is the upper half-plane.
- $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{D} := \mathbb{D}_1$.
- $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$ and $\mathbb{T} := \mathbb{T}_1$.
- $\mathbb{C}^* := \mathbb{C} \smallsetminus \{0\}$ and $\mathbb{D}^* := \mathbb{D} \smallsetminus \{0\}$.
- $\mathbb{A}_{r,s} := \{z \in \mathbb{C} : r < |z| < s\}$.

We assume the reader is familiar with the basic theory of quasiconformal mappings in the plane, as in [**A**] or [**LV**].

2.1. Quasisymmetric maps.

An orientation-preserving homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is called k-*quasisymmetric* if

$$k^{-1} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq k \tag{2.1}$$

for all $x \in \mathbb{R}$ and $t > 0$. It is well-known from the work of Beurling and Ahlfors that this condition is equivalent to f having a quasiconformal extension to the upper half-plane [**A**]. In particular, they show that the map $f_{\rm BA} : \mathbb{H} \to \mathbb{H}$ defined by

$$f_{\rm BA}(x+iy) := \frac{(1+i)}{2y} \left(\int_x^{x+y} f(t)\, dt - i \int_{x-y}^{x} f(t)\, dt \right)$$

is a K-quasiconformal extension of f, where K depends only on k and not on the choice of f. We call $f_{\rm BA}$ the *Beurling-Ahlfors extension* of f. The assignment $f \mapsto f_{\rm BA}$ is readily seen to be equivariant with respect to the action of the real affine group

$$\mathrm{Aut}(\mathbb{C}) \cap \mathrm{Aut}(\mathbb{H}) = \{z \mapsto az + b : a > 0, b \in \mathbb{R}\},$$

i.e.,

$$(\alpha \circ f \circ \beta)_{\rm BA} = \alpha \circ f_{\rm BA} \circ \beta \tag{2.2}$$

for all α, β in this group.

We use the Beurling-Ahlfors extension to define ***standard*** extensions of circle homeomorphisms to disks and annuli as follows. Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism and $g : \mathbb{R} \to \mathbb{R}$ be a lift of f under the covering map $x \mapsto e^{2\pi i x}$. Note that g is unique up to an additive integer and commutes with the unit translation $x \mapsto x+1$. By definition, f is k-quasisymmetric if g is k-quasisymmetric in the sense of (2.1). In this case the extension $g_{\rm BA} : \mathbb{H} \to \mathbb{H}$ is K-quasiconformal for some $K = K(k)$ and by (2.2) commutes with $z \mapsto z+1$, so it descends under the covering map $z \mapsto e^{2\pi i z}$ to a K-quasiconformal homeomorphism $\hat{f} : \mathbb{D} \to \mathbb{D}$ which extends f and fixes the origin.

The following lemma gives a similar construction for the annulus:

Lemma 2.1. *Suppose $f : \partial\mathbb{A}_{r,s} \to \partial\mathbb{A}_{r,s}$ restricts to k-quasisymmetric maps on each of the circles $\mathbb{T}_r$ and $\mathbb{T}_s$, with $f(r) = r$ and $f(s) = s$. Then f extends to a K-quasiconformal homeomorphism $\hat{f} : \mathbb{A}_{r,s} \to \mathbb{A}_{r,s}$, where $K = K(k, s/r)$.*

Proof. After a radially affine stretch, we may assume $r = 1, s = e^{2\pi^2}$ and construct a K-quasiconformal extension of f with $K = K(k)$. Under the covering map from the strip $S := \{z : 0 < \mathrm{Im}(z) < \pi\}$ to $\mathbb{A}_{r,s}$ defined by $z \mapsto e^{-2\pi i z}$, the map f lifts to a homeomorphism $h : \partial S \to \partial S$ which satisfies $h(0) = 0, h(i\pi) = i\pi$, and commutes with $z \mapsto z+1$. Moreover, the restriction of h to the lines $\mathrm{Im}(z) = 0$ and $\mathrm{Im}(z) = \pi$ is k-quasisymmetric. Under the conformal isomorphism $\mathbb{H} \to S$ defined by $z \mapsto \log z$, the map h induces a homeomorphism $g : \mathbb{R} \to \mathbb{R}$ which satisfies $g(0) = 0, g(1) = 1, g(-1) = -1$, and commutes with $z \mapsto ez$. It is not hard to see that g is k'-quasisymmetric for some k' depending on k. The extension $g_{\rm BA} : \mathbb{H} \to \mathbb{H}$ commutes with $z \mapsto ez$ by (2.2), and it is K-quasiconformal for some K depending only on k', hence on k. The induced K-quasiconformal map $\hat{h} : S \to S$ commutes with $z \mapsto z+1$, so it descends to a K-quasiconformal extension $\hat{f} : \mathbb{A}_{r,s} \to \mathbb{A}_{r,s}$, as required. ☐

2.2. **Quasicircles.** A Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ is called a K-*quasicircle* if there is a K-quasiconformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\gamma = \varphi(\mathbb{T})$. We call γ a *quasicircle* if it is a K-quasicircle for some $K \geq 1$.

The following lemma is standard:

Lemma 2.2. *Let $\gamma \subset \widehat{\mathbb{C}}$ be a K-quasicircle, U be a component of $\widehat{\mathbb{C}} \smallsetminus \gamma$, and $\zeta : \mathbb{D} \to U$ be a conformal isomorphism. Then ζ extends to a K^2-quasiconformal map of the sphere.*

Proof. Take a K-quasiconformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\gamma = \varphi(\mathbb{T})$. The reflection $\iota : z \mapsto 1/\overline{z}$ across $\mathbb{T}$ induces a K^2-quasiconformal reflection $j := \varphi \circ \iota \circ \varphi^{-1}$ which fixes γ pointwise. The homeomorphism

$$\hat{\zeta} := \begin{cases} \zeta & \text{inside } \mathbb{D} \\ j \circ \zeta \circ \iota & \text{outside } \mathbb{D} \end{cases}$$

is then a K^2-quasiconformal extension of ζ to the sphere. □

Theorem 2.3. *Let $\zeta : \mathbb{D} \to U \subset \widehat{\mathbb{C}}$ be a conformal isomorphism. Then, the following conditions are equivalent:*

(i) *The boundary ∂U is a quasicircle.*

(ii) *The Jordan curves $\gamma_r := \zeta(\mathbb{T}_r)$ are K-quasicircles for some K independent of $0 < r < 1$.*

Proof. First suppose ∂U is a K-quasicircle. By Lemma 2.2, ζ extends to a K^2-quasiconformal map $\hat{\zeta}$ of the sphere. Since γ_r is the image of $\mathbb{T}$ under $z \mapsto \hat{\zeta}(rz)$, it follows that γ_r is a K^2-quasicircle for every $0 < r < 1$.

Conversely, suppose there is a K such that each γ_r is a K-quasicircle. Without losing generality assume $\lim_{r \to 1} \zeta(r)$ exists and is $\neq \infty$. Take K-quasiconformal maps $\varphi_r : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\gamma_r = \varphi_r(\mathbb{T})$. Pre-compose φ_r with a Möbius map preserving $\mathbb{D}$ to arrange $\varphi_r(0) = \zeta(0)$ and $\varphi_r(1) = \zeta(r)$. The K-quasiconformal maps $z \mapsto (\varphi_r(z) - \zeta(0))/(\zeta(r) - \zeta(0))$ fix 0 and 1, so by compactness there is a sequence $r_n \to 1$ such that φ_{r_n} tends locally uniformly to a K-quasiconformal map $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (the limit of φ_{r_n} cannot be a constant map since that would mean γ_{r_n} converges to a point). It follows that $\partial U = \varphi(\mathbb{T})$ is a K-quasicircle. □

Remark 2.4. In the situation of Theorem 2.3, suppose $\zeta(0) = 0$ and every γ_r is the image of $\mathbb{T}$ under a K-quasiconformal map $\varphi_r : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which fixes 0 and ∞. Then by the above proof the limit $\varphi = \lim_{n \to \infty} \varphi_{r_n}$ will also have 0 and ∞ as fixed points. It follows in particular that the quasicircle $\partial U = \varphi(\mathbb{T})$ does not pass through ∞.

We will need the following geometric characterization of quasicircles in terms of cross-ratios, which is equivalent to Ahlfors's "bounded turning condition" [**A**]. Define the *cross-ratio* of a quadruple (a, b, c, d) of distinct points in $\widehat{\mathbb{C}}$ by

$$\mathbf{Cr}(a, b, c, d) := \frac{(a-b)(c-d)}{(a-c)(b-d)}. \tag{2.3}$$

It is easily checked that $\mathbf{Cr}$ is invariant under the action of the Möbius group; in particular, $0 < \mathbf{Cr}(a, b, c, d) < 1$ whenever the points a, b, c, d lie on a circle (in this cycle order).

Theorem 2.5. *The following conditions on a Jordan curve $\gamma \subset \widehat{\mathbb{C}}$ are equivalent:*

(i) *γ is a K-quasicircle.*

(ii) *There is a constant $M > 0$ such that for every quadruple of distinct points $a, b, c, d \in \gamma$ (in this cyclic order),*

$$|\,\mathbf{Cr}(a, b, c, d)| \leq M.$$

The constants K and M depend only on each other and not on the choice of γ.

2.3. **Linearization of circle maps.** Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism and $g : \mathbb{R} \to \mathbb{R}$ be a lift of f under the covering map $x \mapsto e^{2\pi i x}$. The limit

$$\lim_{n\to\infty} \frac{g^{\circ n}(x)}{n} \qquad (\mathrm{mod}\ \mathbb{Z})$$

exists and is independent of the choice of $x \in \mathbb{R}$ and the lift g. We call this residue class the *rotation number* of f and often identify it with its unique representative in the interval $[0, 1)$. It is a basic invariant of the conjugacy class of f.

Suppose $f : \mathbb{T} \to \mathbb{T}$ has an irrational rotation number $0 < \theta < 1$. We say f is *topologically linearizable* if there exists a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ which conjugates f to the rigid rotation $R_\theta : z \mapsto e^{2\pi i\theta} z$:

$$f \circ h = h \circ R_\theta \qquad \text{on } \mathbb{T}.$$

The linearizing map h is unique if normalized so that $h(1) = 1$. The map f is quasisymmetrically (resp. smoothly, analytically) linearizable if its normalized linearizing map is quasisymmetric (resp. smooth, real-analytic).

An irrational number $0 < \theta < 1$ is *Diophantine of exponent* $\nu \geq 2$ if there is a constant $C > 0$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{C}{q^\nu}$$

for all rational numbers p/q with $q > 0$. We say θ is of *bounded type* if it is Diophantine of exponent $\nu = 2$. Equivalently, θ is bounded type if the integers a_n in the continued fraction expansion

$$\theta = [a_1, a_2, a_3, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

form a bounded sequence.

The basic result on linearization of real-analytic diffeomorphisms is the following theorem of Herman and Yoccoz (see [**H1**] and [**Y**]):

Theorem 2.6 (Herman-Yoccoz). *Every real-analytic circle diffeomorphism with a Diophantine rotation number is analytically linearizable.*

In the presence of critical points, however, the situation is much more subtle. Let us call a quadruple (a, b, c, d) of points in $\mathbb{T}$ *sorted* if they appear in this cyclic order as we go around the circle counterclockwise. The cross-ratio of a sorted quadruple is defined by (2.3) and satisfies $0 < \mathbf{Cr}(a, b, c, d) < 1$. Given an orientation-preserving homeomorphism $f : \mathbb{T} \to \mathbb{T}$ and an interval $I \subset \mathbb{T}$, define the *cross-ratio distortion of f on I* by

$$\mathcal{D}(f, I) := \sup\ \log \frac{\mathbf{Cr}(f(a), f(b), f(c), f(d))}{\mathbf{Cr}(a, b, c, d)},$$

where the supremum is taken over all sorted quadruples (a, b, c, d) of points in I. For a collection $\mathfrak{J}$ of intervals in $\mathbb{T}$, we define the *thickness* $\tau(\mathfrak{J})$ as the maximum number of overlapping intervals in $\mathfrak{J}$. Equivalently,

$$\tau(\mathfrak{J}) = \sup_{\mathbb{T}} \sum_{I \in \mathfrak{J}} \chi_I,$$

where χ_I is the characteristic function of the interval I. Finally, define the *cross-ratio distortion norm* of f by

$$\mathcal{D}(f) := \sup \frac{1}{\tau(\mathfrak{J})} \sum_{I \in \mathfrak{J}} \mathcal{D}(f, I),$$

where the supremum is taken over all collections $\mathfrak{J}$ with finite thickness.

The following theorem of Herman and Swiatek addresses the linearization problem of real-analytic circle homeomorphisms, allowing the presence of critical points (see [**H3**] and [**S**]):

Theorem 2.7 (Herman-Swiatek). *Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism whose rotation number θ is an irrational of bounded type.*

(i) *If the cross-ratio distortion norm $\mathcal{D}(f)$ is finite, then f is k-quasisymmetrically linearizable, where k depends only on $\mathcal{D}(f)$ and θ.*

(ii) *If f is real-analytic, then $\mathcal{D}(f)$ is finite. More precisely, suppose there are constants $\delta > 1$ and $M > 0$ such that f extends holomorphically to the annulus $\delta^{-1} < |z| < \delta$ and satisfies*

$$\left| \frac{zf'(z)}{f(z)} \right| \leq M \qquad \textit{whenever } \delta^{-1} < |z| < \delta. \tag{2.4}$$

Then $\mathcal{D}(f) < C$, where the constant $C > 0$ depends only on δ, M. As a result, f will be k-quasisymmetrically linearizable, with k depending only on θ, δ, M.

Here is how we interpret (2.4): Consider the strip $S := \{z : |\operatorname{Im}(z)| < \log \delta/(2\pi)\}$ and let $g : S \to \mathbb{C}$ be the lift of f under the exponential map $z \mapsto w = e^{2\pi i z}$ which satisfies $0 \leq g(0) < 1$. A simple computation shows that $g'(z) = wf'(w)/f(w)$, so the condition (2.4) translates into the bound $|g'| \leq M$ in S. This, in turn, gives a bound (depending on δ, M) on the size of the 1-periodic function $z \mapsto g(z) - z$ in S, which is essential in the proof of Theorem 2.7.

2.4. **Siegel disks.** Let $0 < \theta < 1$ be an irrational number and f be a non-linear holomorphic map defined in a neighborhood of the origin, with $f(0) = 0$ and $f'(0) = e^{2\pi i \theta}$. We say f is *locally linearizable* at the origin if there exists a holomorphic change of coordinates near 0 which conjugates f to its linear part $R_\theta : z \mapsto e^{2\pi i \theta} z$. The largest neighborhood of 0 in which f is conjugate to R_θ is a simply-connected domain $\Delta = \Delta_f$ called the *Siegel disk* of f centered at 0. Let $\zeta = \zeta_f : \mathbb{D} \to \Delta$ be the unique conformal isomorphism such that $\zeta(0) = 0$ and $\zeta'(0) > 0$. The number $\zeta'(0)$ is called the *conformal radius* of Δ. Applying Schwarz Lemma to $\zeta^{-1} \circ f \circ \zeta$ shows that ζ conjugates f to R_θ:

$$f \circ \zeta = \zeta \circ R_\theta \qquad \text{in } \mathbb{D}.$$

We often refer to ζ as the *linearizing map* of f in Δ.

According to Siegel [**Si**], when θ is Diophantine, every holomorphic map f with $f(0) = 0$ and $f'(0) = e^{2\pi i \theta}$ is locally linearizable at 0. In particular, f has a Siegel disk centered at 0 if the rotation number θ is of bounded type.

The following result, originally due to Ghys [**Gh**], will be used in the proof of the Main Theorem:

Theorem 2.8. *Let $f : \mathbb{C} \to \mathbb{C}$ be a non-linear entire map with $f(0) = 0$ and $f'(0) = e^{2\pi i \theta}$, where $0 < \theta < 1$ is Diophantine. Suppose the Siegel disk boundary $\partial\Delta$ is a Jordan curve in $\mathbb{C}$. Then $\partial\Delta$ contains a critical point of f.*

Proof. Assume there are no critical points on $\partial\Delta$. Then f is univalent in a neighborhood of the closed disk $\overline{\Delta}$. Take a conformal isomorphism $\varphi : \mathbb{C} \smallsetminus \overline{\Delta} \to \mathbb{C} \smallsetminus \overline{\mathbb{D}}$. The map $g := \varphi \circ f \circ \varphi^{-1}$ is well-defined and holomorphic in an outer neighborhood of the unit circle $\mathbb{T}$. By Schwarz Reflection Principle, g extends holomorphically to an annular neighborhood of $\mathbb{T}$. In particular, $g : \mathbb{T} \to \mathbb{T}$ is a real-analytic diffeomorphism and its rotation number is easily seen to be θ. Since θ is assumed Diophantine, Theorem 2.6 shows that g is analytically conjugate to R_θ on $\mathbb{T}$ and hence on a neighborhood of the circle. Pulling this neighborhood back by φ, it follows that f is conjugate to R_θ in an outer neighborhood of $\partial\Delta$. This contradicts the maximality of the Siegel disk Δ. □

Remark 2.9. The assumptions that f is entire and $\partial\Delta$ is a Jordan curve are not essential. In fact, the theorem holds if we only assume that $\partial\Delta$ is a compact subset of the plane on which f acts injectively (see [**H2**] and compare [**P**] and [**Z2**]). The injectivity assumption can be dispensed with when the rotation number is of bounded type [**GS**].

3. The families $\mathcal{E}^{p,q}$ and $\mathcal{E}^{p,q}(\theta)$

3.1. **Generalities.** First consider the family $\mathcal{E}^{p,q}$ of all non-constant entire maps of the form

$$f(z) = P(z)\,\exp(Q(z)), \tag{3.1}$$

where P and Q are polynomials of degrees p and q, respectively. Thus, f is polynomial if $q = 0$ and transcendental if $q > 0$. Counting multiplicities, f has p zeros and $p+q-1$ critical points (the roots of the polynomial equation $P' + PQ' = 0$). Note that the representation (3.1) is not quite unique. In fact, another pair $\hat{P}, \hat{Q}$ represents the same f if and only if $\hat{P} = e^{-c}P$ and $\hat{Q} = Q + c$ for some constant $c \in \mathbb{C}$.

It will be useful to have a simple characterization for the entire maps in the family $\mathcal{E}^{p,q}$. Recall that the ***growth order*** of an entire map $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$\limsup_{r\to+\infty} \frac{\log\log M(f,r)}{\log r},$$

where $M(f,r) := \sup_{|z|=r} |f(z)|$. For example, the growth order of every map in $\mathcal{E}^{p,q}$ is q.

Lemma 3.1. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire map of finite growth order, with p zeros and $p + q - 1$ critical points counting multiplicities. Then $f \in \mathcal{E}^{p,q}$.*

Proof. Let P be a polynomial of degree p with the same zeros of the same multiplicities as f. The singularities of f/P are removable and the resulting entire map is nowhere vanishing. It follows that $f = P\,\exp(Q)$ for some entire function Q.

The growth order of f and f/P are the same, so $\exp(Q)$ must be of finite growth order by the assumption. It easily follows that Q must be a polynomial of some degree d. The number $p + d - 1$ of critical points of f is by the assumption equal to $p + q - 1$. Hence $d = q$ and $f \in \mathcal{E}^{p,q}$, as required. □

Corollary 3.2. *Suppose $g : \mathbb{C} \to \mathbb{C}$ is entire and there are quasiconformal maps $\varphi, \hat{\varphi} : \mathbb{C} \to \mathbb{C}$ which fix 0 such that $\varphi^{-1} \circ g \circ \hat{\varphi} \in \mathcal{E}^{p,q}$. Then $g \in \mathcal{E}^{p,q}$.*

Proof. Clearly g has p zeros and $p+q-1$ critical points counting multiplicities. By Lemma 3.1 it suffices to check that g has finite growth order. Let $f := \varphi^{-1} \circ g \circ \hat{\varphi}$. As quasiconformal maps, φ and $\hat{\varphi}$ satisfy Hölder conditions of the form

$$C_1|z|^{1/K} \leq |\hat{\varphi}(z)| \quad \text{and} \quad |\varphi(z)| \leq C_2|z|^K$$

for large $|z|$, where $C_1, C_2 > 0$ and $K > 1$ are constants. It follows from $\varphi \circ f = g \circ \hat{\varphi}$ that

$$M(g,r) \leq C_2(M(f, C_3 r^K))^K \qquad \text{for large } r,$$

where $C_3 = C_1^{-K}$. This shows that the growth order of g is at most K times that of f, that is at most Kq. □

We will need a few general facts about mapping properties of elements of $\mathcal{E}^{p,q}$. We begin with the following version of the "monodromy theorem," a standard result which is included here for convenience.

Theorem 3.3. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a non-constant entire map, V is a domain containing no asymptotic value of f, and U is a connected component of $f^{-1}(V)$ containing no critical point of f. Then $f : U \to V$ is a covering map.*

Recall that $v \in \mathbb{C}$ is an *asymptotic value* of f if there is a path $\eta : [0,1) \to \mathbb{C}$ such that $\lim_{t\to 1} \eta(t) = \infty$ and $\lim_{t \to 1} f(\eta(t)) = v$.

Proof. The map $f : U \to f(U)$ is a local homeomorphism with the curve lifting property, so it must be a covering. Assume by way of contradiction that $f(U) \neq V$, and choose a path $\eta : [0,1) \to f(U)$ such that $a := \lim_{t\to 1} \eta(t) \in \partial f(U) \cap V$. Let $\hat{\eta} : [0,1) \to U$ be any lift of η. Since $a \in V$ is not an asymptotic value, $\hat{\eta}(t)$ cannot tend to ∞ as $t \to 1$. Hence $\hat{\eta}$ must have a finite accumulation point $\hat{a} \in \overline{U}$ where $f(\hat{a}) = a$. Now if $\hat{a} \in U$ then $a \in f(U)$, and if $\hat{a} \in \partial U$ then $a \in f(\partial U) \subset \partial V$. In either case we reach a contradiction, so $f(U) = V$. □

Corollary 3.4. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a non-constant entire map, V is a domain containing no asymptotic value of f, and U is a connected component of $f^{-1}(V)$ containing at most finitely many critical points of f. Then $f(U) = V$.*

Proof. Let C be the finite (possibly empty) set of critical points of f in U. By Theorem 3.3, $f : U \smallsetminus f^{-1}(f(C)) \to V \smallsetminus f(C)$ is a covering map. In particular, f maps $U \smallsetminus f^{-1}(f(C))$ onto $V \smallsetminus f(C)$, from which it follows that $f(U) = V$. □

Now let $f = P \exp(Q) \in \mathcal{E}^{p,q}$ with $q = \deg Q > 0$. To study the behavior of the transcendental map f near infinity, it will be convenient to introduce the following notion. Since the polynomial Q acts like $z \mapsto z^q$ in suitable coordinates near ∞, there are $2q$ equally spaced rays coming together at ∞ along which $\mathrm{Re}(Q) = 0$. We call these the *neutral directions* of f at infinity. They divide a punctured neighborhood of ∞ into q *positive sectors* in which $\mathrm{Re}(Q) > 0$ interjected with q *negative sectors* in which $\mathrm{Re}(Q) < 0$ (see Fig. 3).

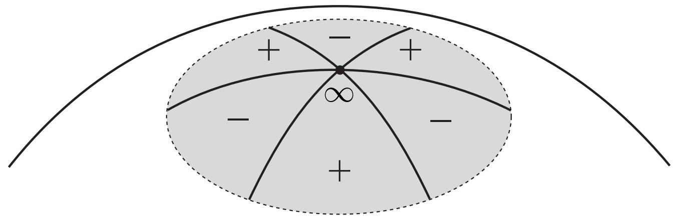

FIGURE 3. *Positive and negative sectors of a map $f \in \mathcal{E}^{p,q}$ near ∞; here $q = 3$.*

Theorem 3.5. *Every $f \in \mathcal{E}^{p,q}$ with $q > 0$ has a unique asymptotic value at 0.*

Proof. Clearly 0 is an asymptotic value. Suppose by way of contradiction that $v \neq 0$ is an asymptotic value and choose a path $\eta : [0, 1) \to \mathbb{C}$ such that $\eta(t) \to \infty$ and $f(\eta(t)) \to v$ as $t \to 1$. When P is constant, this gives $\exp(Q(\eta(t))) \to v/P \neq 0$, which shows that the path $t \mapsto Q(\eta(t))$ has a well-defined limit as $t \to 1$, which is impossible since $\lim_{t\to1} Q(\eta(t)) = \infty$. So let us assume for the rest of the proof that $p = \deg P > 0$.

Since $f(z) \to \infty$ as $z \to \infty$ in each positive sector, we see that $\eta(t)$ must be contained in the closure of a negative sector for all t close to 1. Hence, there is a continuous branch of the path $t \mapsto \log(P(\eta(t)))$ whose imaginary part remains bounded. Since

$$f(\eta(t)) = P(\eta(t)) \exp(Q(\eta(t))) \to v \neq 0$$

as $t \to 1$, it follows that the path

$$t \mapsto \log P(\eta(t)) + Q(\eta(t))$$

has a well-defined limit as $t \to 1$. Hence,

$$\frac{Q(\eta(t))}{\log P(\eta(t))} \to -1 \quad \text{as } t \to 1.$$

This is impossible because as $t \to 1$ the size of the numerator is comparable to $|\eta(t)|^q$ while the denominator, having bounded imaginary part, has size comparable to $\log |P(\eta(t))|$, which in turn is comparable to $\log |\eta(t)|$. $\square$

Corollary 3.6. *Suppose $f \in \mathcal{E}^{p,q}$ with $q > 0$. If $f(0) = 0$, then each iterate $f^{\circ k}$ has a unique asymptotic value at 0.*

3.2. **Covering properties of maps in $\mathcal{E}^{p,q}$.** We continue assuming $f = P \exp(Q) \in \mathcal{E}^{p,q}$ with $q > 0$. Let c be a critical point of f such that $f(c) = 0$. Then c is a root of the equations $P' + PQ' = P = 0$, that is, a common root of P and P'. Hence the number k of such critical points counting multiplicities is at most $p-1$. Since there are $p+q-1$ critical points altogether, it follows that f has $p+q-1-k \geq q > 0$ critical points counting multiplicities that are not mapped to 0. Let $\mathcal{C}$ denote the collection of these critical points and $\mathcal{V} := f(\mathcal{C})$ be the collection of the corresponding critical values in $\mathbb{C}^*$. For each $v \in \mathcal{V}$, take a smooth ray L_v in $\mathbb{C}^*$ from v to ∞, and arrange that the L_v's be disjoint for distinct v's. Each component of $f^{-1}(L_v)$ is either a *non-critical ray*, i.e., a ray from a non-critical preimage of v to ∞, or a "bouquet" of d *critical rays* from a critical preimage c of v to ∞, where $d = d(c) > 1$ is the local degree of f at c. Evidently, a non-critical ray does not separate the plane but each bouquet of d critical rays separates the plane into d connected components. An easy induction on the cardinality of $\mathcal{C}$ then shows that the union of all such bouquets separates the plane into

$$1 + \sum_{c \in \mathcal{C}} (d(c) - 1) = p + q - k$$

connected components. Setting

$$L := \bigcup_{v \in \mathcal{V}} L_v \cup \{0\} \qquad \text{and} \qquad W := \mathbb{C} \smallsetminus f^{-1}(L),$$

it follows that W decomposes into $p+q-k$ unbounded connected components $W_1, \ldots, W_{p+q-k}$. By Theorem 3.3,

$$f : W_j \to \mathbb{C} \smallsetminus L \tag{3.2}$$

is a covering map for each $1 \leq j \leq p+q-k$. As $\mathbb{C} \smallsetminus L$ is conformally isomorphic to the punctured disk, it follows that each W_j is isomorphic to the punctured disk or to the upper half-plane.

- *Case 1.* The degree d of (3.2) is finite. Then W_j is conformally isomorphic to the punctured disk. Setting $\pi_d(z) := z^d$, it follows that there is a covering space isomorphism

$$\begin{tikzcd} W_j \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathbb{C} \smallsetminus \pi_d^{-1}(L) \arrow[d, "\pi_d"] \\ & \mathbb{C} \smallsetminus L \end{tikzcd} \tag{3.3}$$

which induces a homeomorphism between ∂W_j and $\pi_d^{-1}(L)$ except that every critical ray pair in ∂W_j is identified under φ with a single ray in $\pi_d^{-1}(L)$. In particular, W_j is bounded by finitely many rays and is punctured at a unique preimage of 0 where the local degree of f is d.

- *Case 2.* The degree of (3.2) is infinite. Then W_j is conformally isomorphic to the upper half-plane. Setting $E(z) := \exp(z)$, it follows that there is a covering space isomorphism

$$\begin{tikzcd} W_j \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathbb{C} \smallsetminus E^{-1}(L) \arrow[d, "E"] \\ & \mathbb{C} \smallsetminus L \end{tikzcd} \tag{3.4}$$

which induces a homeomorphism between ∂W_j and $E^{-1}(L)$ except that every critical ray pair in ∂W_j is identified under φ with a single ray in $E^{-1}(L)$. In particular, W_j is bounded by countably many rays and does not contain a preimage of 0.

Thus, there is a one-to-one correspondence between the preimages of 0 and the W_j's of type (3.3). Since 0 has $p - k$ distinct preimages, it follows that $p - k$ of the W_j's are of type (3.3) and q of them are of type (3.4).

The above covering space description is used in the proof of the following two lemmas which we will need in §8:

Lemma 3.7. *Let $f \in \mathcal{E}^{p,q}$ with $q > 0$. Suppose V is a simply-connected domain in $\mathbb{C}^*$ with ∂V locally-connected, and U is a connected component of $f^{-1}(V)$. Then $f : U \to V$ is a proper map.*

Proof. By Corollary 3.4 and Theorem 3.5, $f(U) = V$. Since ∂V is locally-connected, we can choose the rays L_v in the above construction so that $V \smallsetminus L$ is still a simply-connected domain. It follows that each component of $\pi_d^{-1}(V \smallsetminus L)$ (resp. $E^{-1}(V \smallsetminus L)$) is simply-connected and maps conformally to $V \smallsetminus L$ under π_d (resp. E). Take a ray R in $\mathbb{C} \smallsetminus \overline{V \cup L}$ from 0 to ∞. The d lifts $\pi_d^{-1}(R)$ together with the origin divide the plane into d sectors each containing precisely one component of $\pi_d^{-1}(V \smallsetminus L)$. Similarly, the lifts $E^{-1}(R)$ divide the plane into countably many strips each containing precisely one component of $E^{-1}(V \smallsetminus L)$. It follows that distinct components of $\pi_d^{-1}(V \smallsetminus L)$ (resp. $E^{-1}(V \smallsetminus L)$) can be separated by a simple arc in $\pi_d^{-1}(\mathbb{C} \smallsetminus \overline{V \cup L})$ (resp. $E^{-1}(\mathbb{C} \smallsetminus \overline{V \cup L})$). Using the fact that for each j the covering map $f : W_j \to \mathbb{C} \smallsetminus L$ satisfies one of the isomorphisms (3.3) or (3.4), we see that the components U_{ij} of $f^{-1}(V \smallsetminus L) \cap W_j$ are simply-connected and $f : U_{ij} \to V \smallsetminus L$ is a conformal isomorphism. Furthermore, by the above remark, any two such components can be separated by a simple arc in W_j which avoids $f^{-1}(V)$.

Now let j be such that $U \cap W_j \neq \emptyset$. We claim that $U \cap W_j = U_{ij}$ for a unique i. To see this, suppose U_{ij} and U_{kj} are both contained in U for some $i \neq k$. Separate U_{ij} from U_{kj} by a simple arc η in W_j which avoids $f^{-1}(V)$. Since U is connected, there is a path in U from a point in U_{ij} to a point in U_{kj}. This path is bound to intersect η, contradicting $\eta \cap f^{-1}(V) = \emptyset$. Thus, we have proved that whenever U meets W_j, the intersection $U \cap W_j$ is simply-connected and $f : U \cap W_j \to V \smallsetminus L$ is a conformal isomorphism. Since there are finitely many of the W_j, it easily follows that $f : U \to V$ must be proper. $\square$

Lemma 3.8. *Let $f \in \mathcal{E}^{p,q}$ with $q > 0$ and η be a Jordan curve which winds around the origin. Then $f^{-1}(\eta)$ has finitely many components. Furthermore,*

(i) *If η avoids the critical values of f, each component of $f^{-1}(\eta)$ is either a Jordan curve or a simple arc both ends of which tend to ∞. In the latter case, each end is eventually in the closure of some negative sector and asymptotic to a neutral direction of f at infinity.*

(ii) *If η does contain critical values of f, each component of $f^{-1}(\eta)$ is of the form cited in (i) except that we must allow finitely many self-intersections at the critical points.*

Proof. Arrange that each ray L_v intersect η in at most one point. The preimages $\pi_d^{-1}(\eta)$ and $E^{-1}(\eta)$ are clearly connected. Pulling back under the covering space isomorphism φ in (3.3) or (3.4) shows that $f^{-1}(\eta) \cap \overline{W_j}$ consists of finitely many components. Since there are finitely many of the W_j, this shows finiteness of the number of components of $f^{-1}(\eta)$.

Now suppose η avoids the critical values of f and $\hat{\eta}$ is a component of $f^{-1}(\eta)$. Then $\hat{\eta}$ is a one-dimensional submanifold of the plane. Since $\hat{\eta}$ is closed in $\mathbb{C}$, it must be a Jordan curve if it is bounded and a simple arc going to ∞ in both directions if it is unbounded. In the latter case, the ends of $\hat{\eta}$ must eventually lie in the closure of a negative sector since $f(z) \to \infty$ as $z \to \infty$ in a positive sector. These ends should be asymptotic to neutral directions, for otherwise their image would tend to 0.

The case where η contains critical values follows from a straightforward modification of the above argument. $\square$

3.3. The family $\mathcal{E}^{p,q}(\theta)$.

Now let $0 < \theta < 1$ be an irrational of bounded type and consider the subfamily $\mathcal{E}^{p,q}(\theta) \subset \mathcal{E}^{p,q}$ of the entire maps f which have a Siegel disk of rotation number θ centered at the origin. The condition $f(0) = 0$ shows $p \geq 1$ and $q \geq 0$, and our assumption that f is non-linear implies $q > 0$ when $p = 1$.

It will be convenient to normalize maps in $\mathcal{E}^{p,q}(\theta)$ by assuming the following:

- Each $f \in \mathcal{E}^{p,q}(\theta)$ is uniquely represented as $f = P \exp(Q)$, where $P(0) = Q(0) = 0$ and $P'(0) = f'(0) = e^{2\pi i\theta}$. This can be achieved by replacing Q by $Q - Q(0)$ and P by $e^{Q(0)}P$.
- For each $f \in \mathcal{E}^{p,q}(\theta)$ the conformal radius of the Siegel disk Δ_f is equal to 1. Let $\zeta_f : \mathbb{D} \to \Delta_f$ the unique linearizing map which satisfies $\zeta_f(0) = 0$ and $\zeta_f'(0) > 0$. For $\alpha \in \mathbb{C}^*$, set $f_\alpha(z) = f(\alpha z)/\alpha$. Since

$$\zeta_{f_\alpha}(z) = \frac{1}{\alpha}\, \zeta_f\Big(\frac{\alpha}{|\alpha|} z\Big),$$

we can always choose a representative f in each linear conjugacy class such that $\zeta_f'(0) = 1$. Any two such representatives will then be conjugate by a rotation.

When the map f is fixed and there is no danger of confusion, we will drop the subscript f from the notations Δ_f, ζ_f, etc.

For $f \in \mathcal{E}^{p,q}(\theta)$ and $0 < r < 1$, we define

$$\begin{aligned} \Delta_r = \Delta_{f,r} &:= \zeta(\mathbb{D}_r) \\ \gamma_r = \gamma_{f,r} &:= \zeta(\mathbb{T}_r) \\ \Omega_r = \Omega_{f,r} &:= \bigcup_{n \geq 0} f^{-n}(\Delta_r). \end{aligned} \tag{3.5}$$

Thus, Δ_r is an invariant subdisk of Δ bounded by the real-analytic invariant curve γ_r, and Ω_r is the smallest totally invariant set containing Δ_r.

4. Main constructions

4.1. **A quasiconformal reflection.** Fix a map $f \in \mathcal{E}^{p,q}(\theta)$ and a radius $0 < r < 1$. Consider the radii

$$0 < a := r^{3/2} < r < b := r^{1/2} < 1$$

and the open f-invariant annuli

$$\begin{aligned} A_a &:= \zeta(\mathbb{A}_{a,r}) \\ A_b &:= \zeta(\mathbb{A}_{r,b}) \\ A &:= \zeta(\mathbb{A}_{a,b}) = A_a \cup \gamma_r \cup A_b \end{aligned}$$

(see Fig. 4). Note that as $r \to 1$, the modulus of A_a, A_b and A tends to zero.

The main construction begins with the choice of an orientation-reversing quasiconformal reflection $I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with the following properties:

- $I|_{\gamma_r} = \mathrm{id}$ and $I(\Delta_r) = \widehat{\mathbb{C}} \smallsetminus \overline{\Delta_r}$;
- $I : \Delta_a \to \widehat{\mathbb{C}} \smallsetminus \overline{\Delta_b}$ is the unique anti-conformal map normalized by the conditions $I(0) = \infty$ and $I(\zeta(a)) = \zeta(b)$.

A priori, these conditions may force I to have a big dilatation (depending on r) inside the annulus A. But, as it turns out, this will not be a cause for concern.

The choice of such I, of course, is far from unique. In order to have something explicit to work with, we will adopt the following construction. Suppose $\hat{\zeta} : \{z : |z| > b\} \to \mathbb{C} \smallsetminus \overline{\Delta_b}$ is the unique conformal isomorphism such that $\hat{\zeta}(b) = \zeta(b)$. By Lemma 2.2, both $\hat{\zeta}$ and the restriction $\zeta : \mathbb{D}_b \to \Delta_b$ have K_b^2-quasiconformal extensions to the sphere, where K_b is the quasicircle constant of the invariant curve

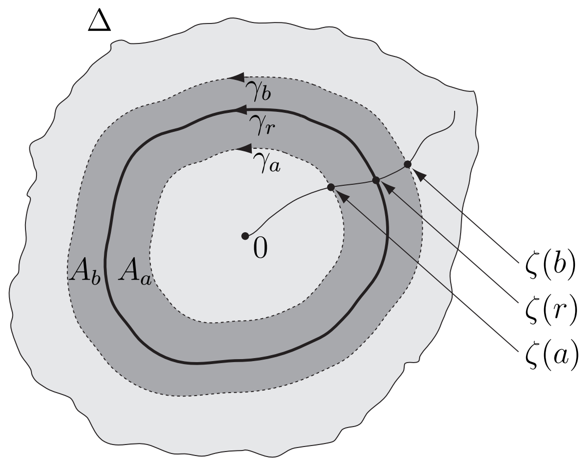

FIGURE 4. *Some invariant curves and annuli in the Siegel disk Δ.*

γ_b. Hence the restriction of $\zeta^{-1} \circ \hat{\zeta}$ to $\mathbb{T}_b$ is k-quasisymmetric with k depending only on K_b. Let $\varphi : \partial\mathbb{A}_{r,b} \to \partial\mathbb{A}_{r,b}$ be the homeomorphism which restricts to the identity on $\mathbb{T}_r$ and to $\zeta^{-1} \circ \hat{\zeta}$ on $\mathbb{T}_b$. Use Lemma 2.1 to extend φ to a K-quasiconformal map $\mathbb{A}_{r,b} \to \mathbb{A}_{r,b}$, where $K = K(K_b, r)$. This allows us to extend $\hat{\zeta}$ to a K-quasiconformal map $\{z : |z| > r\} \to \mathbb{C} \smallsetminus \overline{\Delta_r}$ by setting it equal to $\zeta \circ \varphi$ on $\mathbb{A}_{r,b}$. Denoting by ι the reflection $z \mapsto r^2/\overline{z}$, we can now define $I = \hat{\zeta} \circ \iota \circ \zeta^{-1}$ in Δ_r and set $I = I^{-1}$ elsewhere.

Corollary 4.1. *The maximal dilatation of the quasiconformal reflection I constructed above depends only on r and the quasicircle constant of the invariant curve γ_b.*

Observe that the quasicircle constant of γ_b generally depends on the radius r *as well as* on the map f.

4.2. **Symmetrizing f.** Next we construct a quasiregular dynamics $F : \mathbb{C}^* \to \widehat{\mathbb{C}}$ by symmetrizing f about the invariant curve γ_r using the reflection I:

$$F := \begin{cases} f & \text{outside } \Delta_r \\ I \circ f \circ I & \text{in } \Delta_r \smallsetminus \{0\}. \end{cases}$$

Note that F has a "quasiconformal Herman ring" $\Delta \cap I(\Delta)$ containing the invariant annulus A.

Theorem 4.2. *The map F is*

(i) *holomorphic outside $\overline{\Delta_r} \cap F^{-1}(\overline{A})$;*

(ii) *symmetric about* γ_r *in the sense that* $F \circ I = I \circ F$.

Proof. Outside $\overline{\Delta_r}$, $F = f$ is clearly holomorphic. In the open set $\overline{\Delta_r} \smallsetminus F^{-1}(\overline{A})$, $F = I \circ f \circ I$ is a composition of one holomorphic and two anti-holomorphic maps, hence is holomorphic. This proves (i).

For (ii), observe that if $z \notin \Delta_r$,

$$(F \circ I)(z) = (I \circ f \circ I \circ I)(z) = (I \circ f)(z) = (I \circ F)(z),$$

while if $z \in \Delta_r$,

$$(F \circ I)(z) = (f \circ I)(z) = (I \circ I \circ f \circ I)(z) = (I \circ F)(z). \qquad \square$$

4.3. **Straightening F.** Below we show that the symmetric map F constructed above is quasiconformally conjugate to a holomorphic map.

Theorem 4.3. *There exists a measurable conformal structure μ of bounded dilatation on $\widehat{\mathbb{C}}$ which is invariant under the action of both F and I.*

In general, the dilatation of μ depends on the maximal dilatation of I, and *a priori* it can grow large as $r \to 1$.

Proof. Define μ on A by setting $\mu := \mu_0$ on $A_b \cup \gamma_r$ and $\mu := I^*(\mu_0)$ on A_a. Here μ_0 denotes the standard conformal structure of the plane represented by the zero Beltrami differential. Since F is holomorphic in A_b and $F \circ I = I \circ F$, μ is invariant under $F : A \to A$. Spread μ along the backward orbit of A by using the iterates of F, i.e., define

$$\mu := (F^{\circ n})^*(\mu) \qquad \text{on } F^{-n}(A).$$

On the rest of $\widehat{\mathbb{C}}$, set $\mu = \mu_0$. It is clear from the definition that μ is F-invariant. Using the symmetry relation $F \circ I = I \circ F$ again, we see that the conformal structure $I^*(\mu)$ must also be F-invariant. Since $I^*(\mu) = \mu$ holds in A, it should hold everywhere, which means μ is I-invariant.

Finally, μ has bounded dilatation on A since I is quasiconformal. The first pull-back of μ to $F^{-1}(A) \smallsetminus A$ can increase the dilatation because the branch of F used for pulling back need not be holomorphic. However, all the subsequent pull-backs are taken using the branches of F which are holomorphic by Theorem 4.2, so they will not increase the dilatation further. $\square$

According to the Measurable Riemann Mapping Theorem [**A**], there exists a quasiconformal map $\xi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which solves the Beltrami equation $\xi^*(\mu_0) = \mu$. Moreover, ξ is unique once it is normalized by the conditions

$$\xi(0) = 0, \quad \xi(\infty) = \infty \quad \text{and} \quad \xi(\zeta(r)) = 1.$$

Theorem 4.4. *The homeomorphism ξ*

(i) *is conformal off* Ω_r*, in the sense that* $\overline{\partial}\xi = 0$ *a.e. on the closed set* $\mathbb{C} \smallsetminus \Omega_r$*;*

(ii) *conjugates* I *to the reflection* $\iota : z \mapsto 1/\overline{z}$*;*

(iii) *maps* γ_r *homeomorphically to the unit circle.*

Proof. The forward f-orbit of every point z outside Ω_r is either disjoint from A or else lands in $A_b \cup \gamma_r$. In either case, it follows from the construction that $\mu(z) = 0$, proving (i).

The quasiconformal map $\hat{\xi} := \iota \circ \xi \circ I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ also solves the Beltrami equation $\hat{\xi}^*(\mu_0) = \mu$:

$$\hat{\xi}^*(\mu_0) = (I^* \circ \xi^* \circ \iota^*)(\mu_0) = (I^* \circ \xi^*)(\mu_0) = I^*(\mu) = \mu.$$

Since $\hat{\xi}$ fixes 0 and ∞ and sends $\zeta(r)$ to 1, we have $\hat{\xi} = \xi$ by uniqueness. This proves (ii).

The assertion (iii) follows immediately since each of these Jordan curves is characterized as the fixed point set of the corresponding reflection. □

Now consider the conjugate quasiregular map $G : \mathbb{C}^* \to \widehat{\mathbb{C}}$ defined by

$$G := \xi \circ F \circ \xi^{-1}. \tag{4.1}$$

Theorem 4.5. *The map* G

(i) *is holomorphic;*

(ii) *commutes with the reflection* $\iota : z \mapsto 1/\overline{z}$*, hence preserves the unit circle* $\mathbb{T}$*;*

(iii) *has a Herman ring* $\xi(\Delta \cap I(\Delta))$ *of rotation number* θ *with* $\mathbb{T}$ *as an invariant curve. In particular,* G *restricts to an orientation-preserving real-analytic diffeomorphism of* $\mathbb{T}$ *with rotation number* θ*.*

(iv) *has* $p-1$ *zeros, all in* $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$*, and* $p-1$ *poles, all in* $\mathbb{D}^*$*.*

Recall that the number p in (iv) is the degree of the polynomial P in the representation $f = P \exp(Q) \in \mathcal{E}^{p,q}(\theta)$. If $p = 1$, (iv) is understood as saying that G has no zeros or poles.

Proof. For (i), note that $F^*(\mu) = \mu = \xi^*(\mu_0)$, hence $G^*(\mu_0) = \mu_0$. Thus, as a quasiregular map which preserves the standard conformal structure, G must be holomorphic. Assertion (ii) follows from Theorems 4.2 and 4.4:

$$\begin{aligned} G \circ \iota &= \xi \circ F \circ \xi^{-1} \circ \iota = \xi \circ F \circ I \circ \xi^{-1} \\ &= \xi \circ I \circ F \circ \xi^{-1} = \iota \circ \xi \circ F \circ \xi^{-1} = \iota \circ G. \end{aligned}$$

Assertion (iii) follows easily from the corresponding property of F. For (iv), observe that by the definition of F and the normalization $\xi(0) = 0$,

$$G^{-1}(0) = \xi(F^{-1}(0)) = \xi(f^{-1}(0) \smallsetminus \{0\}) = \xi(P^{-1}(0) \smallsetminus \{0\}).$$

Since 0 is a simple root of P, it follows that G has $p-1$ zeros in $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$ counting multiplicities, and the number of poles is the same by symmetry. $\Box$

4.4. **Surgery.** We now perform a surgery on G to turn it back into an entire function, its Herman ring back into a Siegel disk. The idea is roughly to "cut out" $\mathbb{D}$, "glue in" a quasiconformal Siegel disk instead, and straighten the resulting action in order to realize it as an entire map in $\mathcal{E}^{p,q}(\theta)$.

By Theorem 4.5, $G : \mathbb{T} \to \mathbb{T}$ is a real-analytic diffeomorphism with rotation number θ, which is assumed to be an irrational of bounded type. By Herman-Swiatek's Theorem 2.7, the normalized linearizing map $h : \mathbb{T} \to \mathbb{T}$ of G is quasisymmetric. Let $H : \mathbb{D} \to \mathbb{D}$ be the standard quasiconformal extension of h constructed in §2.1 which satisfies $H(0) = 0, H(1) = 1$. Define the modified quasiregular map $\hat{G} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{G} := \begin{cases} G & \text{outside } \mathbb{D} \\ H^{-1} \circ R_\theta \circ H & \text{in } \mathbb{D}. \end{cases}$$

We claim that $\hat{G}$ admits an invariant conformal structure ν of bounded dilatation. In fact, since R_θ is holomorphic, $\nu := H^*(\mu_0)$ is $\hat{G}$-invariant in $\mathbb{D}$ (as before, μ_0 denotes the standard conformal structure of the plane). We spread ν along the backward orbit of $\mathbb{D}$ by setting

$$\nu := (\hat{G}^{\circ n})^*(\nu) \qquad \text{on } \hat{G}^{-n}(\mathbb{D}).$$

On the rest of $\mathbb{C}$, we set $\nu = \mu_0$. By the construction, ν is $\hat{G}$-invariant. Moreover, since the branches of $\hat{G} = G$ used to spread ν around are all holomorphic, the maximal dilatation of ν on $\mathbb{C}$ is the same as its maximal dilatation on $\mathbb{D}$, which is bounded since H is quasiconformal.

By the Measurable Riemann Mapping Theorem, there is a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$ which fixes the origin and solves the Beltrami equation $\psi^*(\mu_0) = \nu$. Consider the conjugate map $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g := \psi \circ \hat{G} \circ \psi^{-1}.$$

Since $\hat{G}^*(\nu) = \nu = \psi^*(\mu_0)$, the definition of g shows that $g^*(\mu_0) = \mu_0$, which means g is an entire function. It clearly has a Siegel disk Δ_g of rotation number θ centered at the origin which contains $\psi(\mathbb{D})$ as a proper invariant subdisk.

To make g unique, we normalize ψ in the following way: Since both H and ψ pull μ_0 back to ν on $\mathbb{D}$, the composition $\psi \circ H^{-1}$ is conformal. Hence, we can choose ψ as the unique quasiconformal solution of $\psi^*(\mu_0) = \nu$ which satisfies

$$\psi(0) = 0 \qquad \text{and} \qquad (\psi \circ H^{-1})'(0) = r. \tag{4.2}$$

Theorem 4.6. *The quasiconformal map $\varphi := \psi \circ \xi : \mathbb{C} \to \mathbb{C}$ has the following properties:*

(i) $\varphi \circ f = g \circ \varphi$ *off* $\Delta_{f,r}$.

(ii) φ *is conformal off* $\Omega_{f,r}$.

(iii) $\varphi(\gamma_{f,r}) = \gamma_{g,r}$.

(iv) $\varphi = \zeta_g \circ \zeta_f^{-1}$ *on* $\gamma_{f,r}$.

Proof. For (i), simply note that on $\mathbb{C} \smallsetminus \Delta_{f,r}$,

$$\begin{aligned}\varphi \circ f &= \psi \circ \xi \circ f \\ &= \psi \circ \xi \circ F \\ &= \psi \circ G \circ \xi \\ &= \psi \circ \hat{G} \circ \xi \\ &= g \circ \psi \circ \xi = g \circ \varphi,\end{aligned}$$

where the fourth equality holds since by Theorem 4.4, ξ maps the complement of $\Delta_{f,r}$ to the complement of $\mathbb{D}$.

Next, ξ is conformal off $\Omega_{f,r}$ by Theorem 4.4 and ψ is conformal off the image $\xi(\Omega_{f,r}) = \bigcup_{n\geq 0} \hat{G}^{-n}(\mathbb{D})$ by the construction of ν. This proves (ii).

Since $\xi(\gamma_{f,r}) = \mathbb{T}$, (iii) is equivalent to showing that $\psi(\mathbb{T}) = \gamma_{g,r}$. Observe that $\psi(\mathbb{T})$ is a g-invariant curve in the Siegel disk Δ_g, hence $\psi(\mathbb{T}) = \gamma_{g,s}$ for some $0 < s < 1$. Since the annulus $\Delta_f \smallsetminus \overline{\Delta_{f,r}}$ is disjoint from $\Omega_{f,r}$, part (ii) shows that $\varphi : \Delta_f \smallsetminus \overline{\Delta_{f,r}} \to \Delta_g \smallsetminus \overline{\Delta_{g,s}}$ is a conformal isomorphism. Hence the two annuli have the same modulus and $r = s$.

Finally, the composition $z \mapsto (\psi \circ H^{-1})(z/r)$ maps $\mathbb{D}_r$ conformally to $\Delta_{g,r}$, fixes the origin and has derivative 1 there by (4.2). The linearizing map ζ_g has the same properties, so by uniqueness $\zeta_g(z) = (\psi \circ H^{-1})(z/r)$ whenever $|z| \leq r$. On the other hand, $(1/r)(\xi \circ \zeta_f)^{-1} : \mathbb{T} \to \mathbb{T}$ conjugates G to R_θ and fixes 1. By uniqueness, $(1/r)(\xi \circ \zeta_f)^{-1} = h = H$ on the unit circle. It follows that when $|z| = r$,

$$(\varphi \circ \zeta_f)(z) = (\psi \circ \xi \circ \zeta_f)(z) = (\psi \circ H^{-1})(z/r) = \zeta_g(z),$$

which proves (iv). $\square$

For future reference, let us record the following fact which was established in the course of the above proof:

Corollary 4.7. $\zeta_g(z) = (\psi \circ H^{-1})(z/r)$ *whenever* $|z| \leq r$. *In particular, the conformal radius of* Δ_g *is* 1.

Theorem 4.8. $g \in \mathcal{E}^{p,q}(\theta)$.

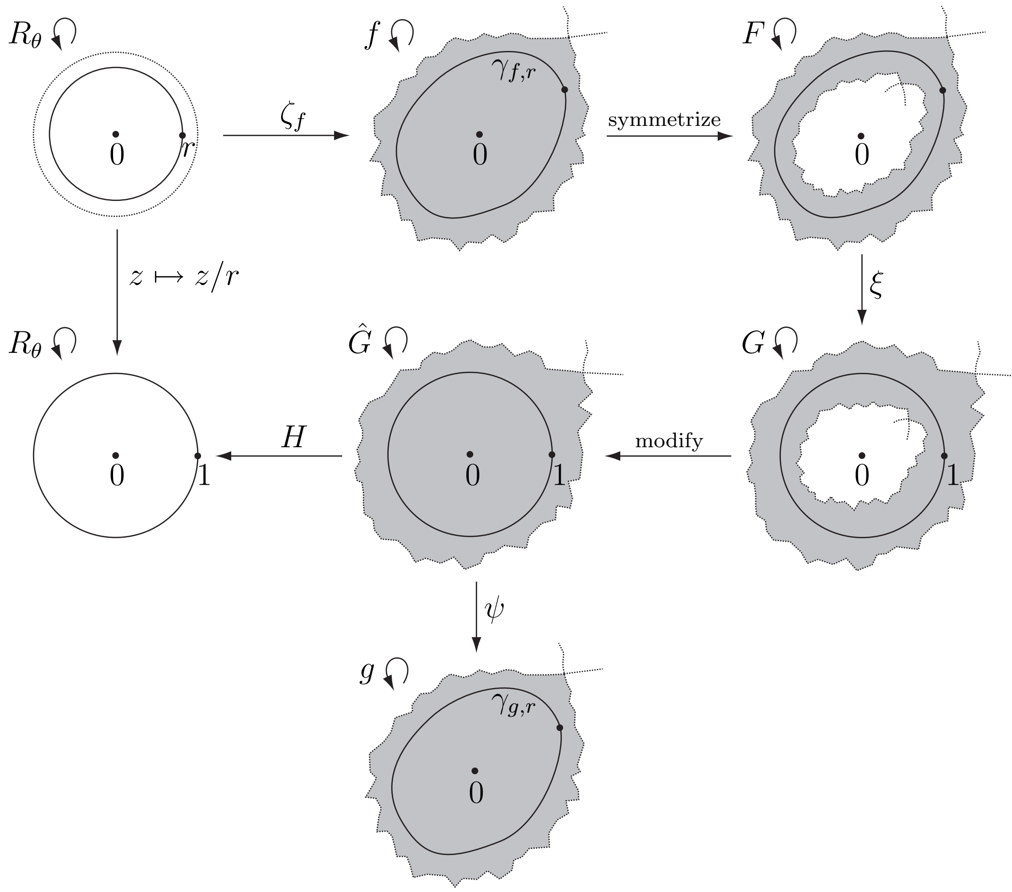

FIGURE 5. *The construction of the surgery map* $\mathcal{S}_r : f \mapsto g$.

Proof. Define

$$\hat{\varphi} := \begin{cases} \varphi & \text{off } \Delta_{f,r} \\ g^{-1} \circ \varphi \circ f & \text{on } \Delta_{f,r}, \end{cases}$$

where g^{-1} refers to the branch of the inverse mapping Δ_g to itself. Evidently $\hat{\varphi}$ is quasiconformal and $\varphi \circ f = g \circ \hat{\varphi}$. It follows from Corollary 3.2 that $g \in \mathcal{E}^{p,q}$. Since g has a Siegel disk of rotation number θ and conformal radius 1 centered at 0, we have $g \in \mathcal{E}^{p,q}(\theta)$. $\square$

Remark 4.9. The map φ is not a conjugacy between f and g inside $\Delta_{f,r}$ unless the extension H of h is chosen so that

$$H = \frac{1}{r}(\xi \circ \zeta_f)^{-1} \qquad \text{in } \mathbb{D}.$$

The reason we did not choose this extension is the dilatation issue: *a priori*, the maximal dilatation of ξ, hence that of $(1/r)(\xi\circ\zeta_f)^{-1}$, depends on r while our argument is heavily based on the fact that there is a quasiconformal extension H whose maximal dilatation is independent of r (Corollary 5.7 below).

Definition 4.10. Let $0 < r < 1$. The *surgery map* $\mathcal{S}_r : \mathcal{E}^{p,q}(\theta) \to \mathcal{E}^{p,q}(\theta)$ is the one which assigns to each f the entire function g constructed above.

Fig. 5 summarizes various steps in the construction of the surgery map $\mathcal{S}_r$.

5. A priori estimate of the dilatation

Let $f \in \mathcal{E}^{p,q}(\theta)$ and let $g := \mathcal{S}_r(f) \in \mathcal{E}^{p,q}(\theta)$ be the result of surgery on f, as described in §4. In this section we prove that the invariant curve $\gamma_{g,r}$ is a K-quasicircle for some $K > 1$ independent of the choice of f and r (Corollary 5.8). This uniformity will be the essential ingredient of the proof of the Main Theorem.

5.1. **The explicit form of G.** Consider the holomorphic map $G : \mathbb{C}^* \to \widehat{\mathbb{C}}$ constructed in §4.3. By Theorem 4.5, G has $p-1$ zeros $\{z_1, \ldots, z_{p-1}\}$, where $|z_j| > 1$ and each root is repeated according to its multiplicity. By symmetry, there are $p-1$ poles at $\{1/\overline{z_1}, \ldots, 1/\overline{z_{p-1}}\}$. Consider the finite Blaschke product

$$B(z) := \prod_{j=1}^{p-1} \left(\frac{z - z_j}{1 - \overline{z_j}\, z} \right) \tag{5.1}$$

which has the same zeros and poles of the same multiplicities as G. When $p = 1$, G has no zeros or poles and we agree to set $B = 1$. In either case, the quotient $S(z) := G(z)/B(z)$ extends to a holomorphic map in $\mathbb{C}^*$ without zeros or poles.

Lemma 5.1. *The map $S : \mathbb{C}^* \to \mathbb{C}^*$ has the form*

$$S(z) = \lambda\, z^n \exp(\alpha(z) - \overline{\alpha(1/\overline{z})}),$$

where $|\lambda| = 1$, n is an integer, and $\alpha : \mathbb{C} \to \mathbb{C}$ is an entire function with $\alpha(0) = 0$.

Proof. As a holomorphic map $\mathbb{C}^* \to \mathbb{C}^*$, S has a *unique* representation

$$S(z) = \lambda\, z^n \exp(\alpha(z) + \beta(1/z)), \tag{5.2}$$

where $\lambda \neq 0$, n is an integer, and α, β are entire functions with $\alpha(0) = \beta(0) = 0$. Since G and B commute with the reflection $z \mapsto 1/\overline{z}$, so does their ratio S. Imposing this condition on the representation (5.2), we obtain

$$\lambda\, z^n \exp(\alpha(z) + \beta(1/z)) = \frac{1}{\overline{\lambda}}\, z^n \exp(-\overline{\alpha(1/\overline{z})} - \overline{\beta(\overline{z})})$$

for all z. Hence, by uniqueness,

$$|\lambda| = 1 \qquad \text{and} \qquad \beta(z) = -\overline{\alpha(\overline{z})}. \qquad \square$$

Lemma 5.2. *The exponent n in Lemma 5.1 is equal to p.*

Proof. Apply the Argument Principle to the function

$$G(z) = B(z)S(z) = \lambda\, z^n\, B(z)\, \exp(\alpha(z) - \overline{\alpha(1/\overline{z})})$$

on the unit circle:

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{G'(z)}{G(z)}\, dz = n + \frac{1}{2\pi i}\int_{|z|=1} \frac{B'(z)}{B(z)}\, dz + \frac{1}{2\pi i}\int_{|z|=1} \frac{d}{dz}(\alpha(z) - \overline{\alpha(1/\overline{z})})\, dz.$$

The left side is equal to 1 since $G : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving homeomorphism. The middle term on the right is $-(p-1)$ since the Blaschke product B has $p-1$ poles and no zeros in $\mathbb{D}$. The term on the far right is zero since the integrand has a holomorphic primitive in a neighborhood of $\mathbb{T}$. Thus, $n = p$ as required. $\square$

Lemma 5.3. *The entire function α of Lemma 5.1 is a polynomial of degree q.*

Proof. When $q = 0$, the map $F = f$ is a polynomial in a neighborhood of infinity, so ∞ is a pole of G. In this case α vanishes identically and the lemma holds. Let us then assume $q > 0$. Since $F = f$ in a neighborhood of infinity, the growth order of F is q, hence the quasiconformally conjugate map G must have finite positive growth order (compare the proof of Corollary 3.2). It follows that $\exp(\alpha)$ is an entire function of finite order, so α is a polynomial of some degree $d > 0$. The number $2(p+q-1)$ of critical points of F must match the number $2(p+d-1)$ for G, hence $d = q$. $\square$

Corollary 5.4. *The holomorphic map $G : \mathbb{C}^* \to \widehat{\mathbb{C}}$ has the form*

$$G(z) = \lambda\, z^p\, B(z)\, \exp(\alpha(z) - \overline{\alpha(1/\overline{z})}), \tag{5.3}$$

where $|\lambda| = 1$, B is a degree $p-1$ Blaschke product as in (5.1) *with all zeros in $\mathbb{C} \smallsetminus \overline{\mathbb{D}}$ (constant function* 1 *if $p = 1$) and α is a polynomial of degree q with $\alpha(0) = 0$.*

5.2. **Linearizing G on the unit circle.** The restriction $G : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving real-analytic diffeomorphism of bounded type rotation number θ. By Theorem 2.7, the normalized linearizing map $h : \mathbb{T} \to \mathbb{T}$ of G is k-quasisymmetric. Moreover, k is bounded by a constant which depends only on θ, the modulus of an annular neighborhood of $\mathbb{T}$ which stays away from the zeros and poles of G, and the size of $zG'(z)/G(z)$ on such an annulus. We begin by estimating how close the poles (equivalently zeros) of the Blaschke product B in (5.3) can be to the unit circle.

Theorem 5.5. *Let $p > 1$ so the Blaschke product B in* (5.3) *is non-constant. There exists a constant $\lambda = \lambda(p, q) > 1$ such that the zeros $\{z_j\}$ of B satisfy $|z_j| > \lambda$ for all $1 \leq j \leq p-1$.*

As the proof will show, one can take $\lambda = (2pq + q + 1)/(2pq + q - 1)$.

Proof. First assume $q > 0$ and take an integer $1 \leq k \leq q$. Set $R(z) := \alpha(z) - \overline{\alpha(1/\overline{z})}$. Logarithmic differentiation of

$$G(z) = \lambda\, z^p\, B(z)\, \exp(R(z))$$

yields

$$\frac{z^k G'(z)}{G(z)} = p\, z^{k-1} + \frac{z^k B'(z)}{B(z)} + z^k R'(z). \tag{5.4}$$

Integrating over the unit circle, we obtain

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{z^k G'(z)}{G(z)}\, dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{z^k B'(z)}{B(z)}\, dz + \frac{1}{2\pi i} \int_{|z|=1} z^k R'(z)\, dz. \tag{5.5}$$

The Argument Principle applied to the formula of B in (5.1) yields

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{z^k B'(z)}{B(z)}\, dz = -\sum_{j=1}^{p-1} \frac{1}{\overline{z_j}^k}. \tag{5.6}$$

To compute the integral of $z^k R'(z)$ over the unit circle, let $\alpha(z) = a_1\, z + \cdots + a_q\, z^q$, so

$$R(z) = -\frac{\overline{a_q}}{z^q} - \cdots - \frac{\overline{a_1}}{z} + a_1\, z + \cdots + a_q\, z^q,$$

and

$$z^k R'(z) = q\, \overline{a_q}\, z^{k-q-1} + \cdots + \overline{a_1}\, z^{k-2} + a_1\, z^k + \cdots + q\, a_q\, z^{k+q-1}. \tag{5.7}$$

Thus,

$$\frac{1}{2\pi i} \int_{|z|=1} z^k R'(z)\, dz = k\, \overline{a_k}. \tag{5.8}$$

Substituting (5.6) and (5.8) into (5.5) and using the fact that $|z_i| > 1$, we obtain

$$k\, |a_k| = \left| \frac{1}{2\pi i} \int_{|z|=1} \frac{z^k G'(z)}{G(z)}\, dz + \sum_{j=1}^{p-1} \frac{1}{\overline{z_j}^k} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |G'(e^{it})|\, dt + p - 1. \tag{5.9}$$

Since $G : \mathbb{T} \to \mathbb{T}$ is an orientation-preserving diffeomorphism, we have $zG'(z)/G(z) = d(\log G(z))/d(\log z) > 0$ on the unit circle. This implies

$$\frac{zG'(z)}{G(z)} = \left| \frac{zG'(z)}{G(z)} \right| = |G'(z)| \qquad \text{whenever } |z| = 1. \tag{5.10}$$

Hence, by another application of the Argument Principle,

$$\frac{1}{2\pi} \int_0^{2\pi} |G'(e^{it})|\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} G'(e^{it})}{G(e^{it})}\, dt = \frac{1}{2\pi i} \int_{|z|=1} \frac{G'(z)}{G(z)}\, dz = 1. \tag{5.11}$$

Putting (5.9) and (5.11) together, we obtain the estimate

$$k\, |a_k| \leq p \qquad \text{for all } \ 1 \leq k \leq q. \tag{5.12}$$

This immediately gives an L^∞ estimate for $zR'(z) = 2\,\mathrm{Re}(z\alpha'(z))$ on the unit circle. In fact, by (5.7) with $k=1$ and (5.12),

$$\sup_{|z|=1} |zR'(z)| \leq 2 \sum_{j=1}^{q} j\,|a_j| \leq 2\,p\,q. \tag{5.13}$$

This, in turn, allows an L^∞ estimate for the logarithmic derivative $zB'(z)/B(z)$ on the unit circle: Start with (5.4) with $k=1$:

$$\frac{zG'(z)}{G(z)} = p + \frac{zB'(z)}{B(z)} + zR'(z).$$

On the unit circle, each term in this identity is real, with the left side being positive by (5.10) and the absolute value of the term on the far right being bounded by $2pq$ by (5.13). Hence,

$$-\frac{zB'(z)}{B(z)} \leq p\,(2q+1) \qquad \text{whenever } |z|=1.$$

A brief computation using (5.1) shows that

$$-\frac{zB'(z)}{B(z)} = \sum_{j=1}^{p-1} \frac{z(|z_j|^2-1)}{(z-z_j)(1-\overline{z_j}z)}, \tag{5.14}$$

so

$$-\frac{zB'(z)}{B(z)} = \sum_{j=1}^{p-1} \frac{|z_j|^2-1}{|z-z_j|^2} \qquad \text{whenever } |z|=1.$$

It follows that for each $1 \leq j \leq p-1$,

$$\frac{|z_j|+1}{|z_j|-1} = \sup_{|z|=1} \frac{|z_j|^2-1}{|z-z_j|^2} \leq \sup_{|z|=1} \frac{-zB'(z)}{B(z)} \leq p\,(2q+1).$$

This gives $|z_j| \geq \lambda$, with

$$\lambda := \frac{2pq+p+1}{2pq+p-1} > 1.$$

In the polynomial case where $q=0$, the rational function R vanishes identically, and the same argument shows that $\lambda = (p+1)/(p-1)$ will work. $\square$

Theorem 5.6. *There exist constants $\delta = \delta(p,q) > 1$ and $M = M(p,q) > 0$ such that the map G in* (5.3) *has no zeros or poles in the annulus $\delta^{-1} < |z| < \delta$, and satisfies*

$$\left| \frac{zG'(z)}{G(z)} \right| \leq M \qquad \textit{whenever } \delta^{-1} < |z| < \delta.$$

Proof. Let $\lambda = \lambda(p,q)$ be the constant given by Theorem 5.5. Set $\delta := \sqrt{\lambda}$ if $p > 1$ and $\delta := 2$ if $p = 1$ (in which case the Blaschke product B in (5.3) is identically 1). By Theorem 5.5, G has no zeros or poles in the annulus $\delta^{-1} < |z| < \delta$. To obtain the bound M, first assume $p > 1, q > 0$. By Theorem 5.5 the zeros $\{z_j\}$ of B satisfy $|z_j| \geq \delta^2$. Hence, when $\delta^{-1} < |z| < \delta$,

$$\begin{aligned} \left| \frac{z(|z_j|^2 - 1)}{(z - z_j)(1 - \overline{z_j} z)} \right| &= \frac{|z_j| - 1}{|z - z_j|} \cdot \frac{|z_j| + 1}{|z^{-1} - \overline{z_j}|} \\ &\leq \frac{|z_j| - 1}{|z_j| - \delta} \cdot \frac{|z_j| + 1}{|z_j| - \delta} \\ &\leq \frac{\delta^2 - 1}{\delta^2 - \delta} \cdot \frac{\delta^2 + 1}{\delta^2 - \delta} \\ &= \frac{\delta^4 - 1}{\delta^2(\delta - 1)^2}. \end{aligned}$$

It follows from (5.14) that

$$\sup_{\delta^{-1} < |z| < \delta} \left| \frac{zB'(z)}{B(z)} \right| \leq M_1 := \frac{(p-1)(\delta^4 - 1)}{\delta^2(\delta - 1)^2}. \tag{5.15}$$

On the other hand, (5.7) and (5.12) together show that

$$\sup_{\delta^{-1} < |z| < \delta} |zR'(z)| \leq M_2 := 2p \sum_{j=1}^{q} \delta^j = \frac{2p\,(\delta^{q+1} - \delta)}{\delta - 1}. \tag{5.16}$$

Now (5.4) with $k = 1$ shows that $|zG'(z)/G(z)|$ is bounded by $M := p + M_1 + M_2$ on the annulus $\delta^{-1} < |z| < \delta$.

In the polynomial case $p > 1, q = 0$ the rational function R is identically zero and the above argument shows that we can take $M := p + M_1$. In the case $p = 1, q > 0$ the Blaschke product B is identically 1 and we can take $M := 1 + M_2$. □

Herman-Swiatek's Theorem 2.7 now implies:

Corollary 5.7. *The normalized linearizing map $h : \mathbb{T} \to \mathbb{T}$ of G is k-quasisymmetric for a constant k depending only on p, q, θ. Hence, its standard extension $H : \mathbb{D} \to \mathbb{D}$ is K-quasiconformal, where K depends only on p, q, θ.*

Corollary 5.8. *Suppose $f \in \mathcal{E}^{p,q}(\theta)$ and $g := \mathcal{S}_r(f)$. Then, the g-invariant curve $\gamma_{g,r}$ is a K-quasicircle for some K which depends only on p, q, θ.*

Proof. By Theorem 4.6, $\gamma_{g,r} = \psi(\mathbb{T})$, where the maximal dilatation of the quasiconformal map ψ is the same as that of the standard extension $H : \mathbb{D} \to \mathbb{D}$. Hence the result follows from Corollary 5.7. □

6. Maps with no free capture spot

Ideally, one would hope that the surgery on $f \in \mathcal{E}^{p,q}(\theta)$ as described in §4 would produce an entire map $\mathcal{S}_r(f)$ which is conformally conjugate to f. However, this type of "rigidity" for a general f is wishful thinking. In reality, f and $\mathcal{S}_r(f)$ may not be even topologically conjugate. The problem arises when f has critical orbits which hit its Siegel disk Δ_f.

6.1. **Captured critical points.** A critical point c of $f \in \mathcal{E}^{p,q}(\theta)$ is said to be *captured* by Δ_f if its forward orbit eventually hits Δ_f. In this case, there is a smallest integer $n \geq 1$ such that $\hat{c} := f^{\circ n}(c) \in \Delta_f$. We call $\hat{c}$ a *capture spot* of f in Δ_f; if $\hat{c} \neq 0$, we call it a *free* capture spot. In this terminology, f has no free capture spot if the forward orbit of each critical point of f is either disjoint from Δ_f or lands directly at the fixed point 0.

Recall that $\zeta_f : \mathbb{D} \to \Delta_f$ is the unique linearizing map of f which is normalized so that $\zeta_f'(0) = 1$. By the *conformal position* of $z \in \Delta_f$ is meant the point $\zeta_f^{-1}(z)$ in the unit disk.

6.2. **Rigidity.** The following theorem shows that the conformal positions of the capture spots are the only obstructions to promoting a quasiconformal conjugacy to a conformal one along the backward orbit of the Siegel disk.

Theorem 6.1. *Suppose $f, g \in \mathcal{E}^{p,q}(\theta)$, $0 < r < 1$ and $\varphi : \mathbb{C} \to \mathbb{C}$ is a quasiconformal map such that*

(i) $\varphi \circ f = g \circ \varphi$ *on* $\mathbb{C} \smallsetminus \Delta_{f,r}$.

(ii) $\overline{\partial}\varphi = 0$ *a.e. on* $\mathbb{C} \smallsetminus \Omega_{f,r}$.

(iii) $\varphi = \zeta_g \circ \zeta_f^{-1}$ *on* $\gamma_{f,r}$.

Let $\{\hat{c}_1, \ldots, \hat{c}_m\}$ be the capture spots of f in $\Delta_{f,r}$ and $\{\hat{e}_1, \ldots, \hat{e}_m\}$ be the corresponding capture spots of g in $\Delta_{g,r}$. If $\hat{c}_j$ and $\hat{e}_j$ have the same conformal position for each j, then $f = g$.

The capture spot $\hat{e}_j$ "corresponds" to $\hat{c}_j$ in the following sense: if $\hat{c}_j = f^{\circ n}(c_j)$ for a critical point c_j of f, then $\hat{e}_j = g^{\circ n}(e_j)$, where $e_j = \varphi(c_j)$.

Proof. We will modify φ along $\Omega_{f,r}$ in order to promote it to a conformal conjugacy Φ between f and g. Set $\Phi := \varphi$ on $\mathbb{C} \smallsetminus \Omega_{f,r}$ and define

$$\Phi := \zeta_g \circ \zeta_f^{-1} \qquad \text{in } \Delta_{f,r}. \tag{6.1}$$

Clearly $\Phi : \Delta_{f,r} \to \Delta_{g,r}$ is a conformal conjugacy between f and g, which, by the condition (iii), is continuous along the invariant curve $\gamma_{f,r}$.

We extend Φ to the remaining part of $\Omega_{f,r}$ as follows. For each non-zero $\hat{c}_j$, consider the radial segment in $\mathbb{D}$ from $\zeta_f^{-1}(\hat{c}_j)$ out to the boundary $\mathbb{T}$ and let J be the union of all such segments together with the segment $[0,1] \subset \mathbb{R}$. Set $L := \zeta_f(J)$. Do the same for g, i.e., consider the radial segments from $\zeta_g^{-1}(\hat{e}_j)$ to $\mathbb{T}$ for each non-zero $\hat{e}_j$, let J' be the union of all such segments together with $[0,1]$, and set $L' := \zeta_g(J')$. Since $\hat{c}_j$ and $\hat{e}_j$ have the same conformal position for each j, we have $J = J'$ and so $L' = \Phi(L)$.

Now let $n \geq 1$ and U be a connected component of $f^{-n}(\Delta_{f,r}) \smallsetminus f^{-n+1}(\Delta_{f,r})$. The slit disk $\Delta_{f,r} \smallsetminus L$ is simply-connected and contains no critical or, by Corollary 3.6, asymptotic value of the iterate $f^{\circ n} : U \to \Delta_{f,r}$. It follows from Theorem 3.3 that the components $\{V_i\}$ of $U \smallsetminus f^{-n}(L)$ are all simply-connected and $f^{\circ n} : V_i \to \Delta_{f,r} \smallsetminus L$ is a conformal isomorphism for each i. Let $U' := \varphi(U)$ and denote by $\{V_i'\}$ the components of $U' \smallsetminus g^{-n}(L')$, where the labeling is chosen so that $\partial V_i' \cap \partial U' = \varphi(\partial V_i \cap \partial U) = \Phi(\partial V_i \cap \partial U)$. Then, by the same reasoning, $g^{\circ n} : V_i' \to \Delta_{g,r} \smallsetminus L'$ is a conformal isomorphism for each i. Since $\Phi(L) = L'$, we can define $\Phi : V_i \to V_i'$ unambiguously by

$$\Phi := g^{-n} \circ \Phi \circ f^{\circ n}.$$

Putting these partially defined maps $V_i \to V_i'$ together, we obtain a homeomorphism $\Phi : U \smallsetminus f^{-n}(L) \to U' \smallsetminus g^{-n}(L')$. Using continuity of Φ along L and $\gamma_{f,r}$, it is easily seen that Φ extends to a homeomorphism $U \to U'$ which is compatible with the boundary map $\Phi = \varphi : \partial U \to \partial U'$. Moreover, since this homeomorphism is conformal off the removable set $f^{-n}(L) \cap U$ of analytic arcs, it must be conformal.

Repeating this process over all components of $f^{-n}(\Delta_{f,r}) \smallsetminus f^{-n+1}(\Delta_{f,r})$ for all $n \geq 1$, we obtain a global conjugacy $\Phi : \mathbb{C} \to \mathbb{C}$ between f and g which is conformal in $\Omega_{f,r}$ and coincides with φ on $\mathbb{C} \smallsetminus \Omega_{f,r}$. To show Φ is conformal everywhere, we can for example invoke the following well-known result in quasiconformal theory (see [**B**] or [**DH**]):

Bers Sewing Lemma. *Suppose $E \subset \mathbb{C}$ is closed, U and V are open neighborhoods of E, and $\varphi : U \to \varphi(U)$ and $\Phi : V \to \Phi(V)$ are homeomorphisms such that*
- *φ is K_1-quasiconformal;*
- *$\Phi|_{V \smallsetminus E}$ is K_2-quasiconformal;*
- *$\varphi = \Phi$ on ∂E.*

Then the map

$$\varphi \amalg \Phi := \begin{cases} \varphi & \text{on } E \\ \Phi & \text{on } V \smallsetminus E \end{cases}$$

is $\max\{K_1, K_2\}$-quasiconformal in V and $\overline{\partial}(\varphi \amalg \Phi) = \overline{\partial}\varphi$ a.e. on E.

Applying this lemma to our maps φ, Φ with $U = V = \mathbb{C}$ and $E = \mathbb{C} \smallsetminus \Omega_{f,r}$, we see that $\Phi = \varphi \amalg \Phi$ is quasiconformal, with $\overline{\partial}\Phi = 0$ a.e. on $\mathbb{C} \smallsetminus \Omega_{f,r}$ by (ii) and everywhere in $\Omega_{f,r}$ by conformality. Thus, Φ is a 1-quasiconformal map of the plane; as such it is a conformal automorphism. As Φ fixes the origin, it must have the form

$\Phi(z) = \alpha z$ for some $\alpha \in \mathbb{C}^*$. Since the conformal radius of both Δ_f and Δ_g is 1, we obtain $\alpha = \Phi'(0) = \zeta_g'(0)/\zeta_f'(0) = 1$. $\square$

Corollary 6.2. *Suppose $f \in \mathcal{E}^{p,q}(\theta)$ has no free capture spot in $\Delta_{f,r}$ for some $0 < r < 1$. Then $\mathcal{S}_r(f) = f$.*

Proof. Apply Theorem 6.1 to f, $g := \mathcal{S}_r(f)$, and the quasiconformal map φ given by Theorem 4.6. $\square$

Proof of the Main Theorem when f has no free capture spot. By Corollary 5.8 and Corollary 6.2, for every $0 < r < 1$ the f-invariant curve $\gamma_{f,r}$ is a K-quasicircle for some K depending only on p, q, θ, hence independent of r. Hence $\partial\Delta_f$ is a quasicircle by Theorem 2.3 and contains a critical point of f by Theorem 2.8. $\square$

7. Maps with one free capture spot

7.1. **A one-dimensional deformation space.** Now consider the case where $f \in \mathcal{E}^{p,q}(\theta)$ has precisely one free capture spot. Recall that this means there is a point $\omega \in \Delta_f \smallsetminus \{0\}$ such that the forward orbit of every captured critical point of f hits Δ_f for the first time at ω or at the fixed point 0.

Theorem 7.1. *For each $t \in \mathbb{D}^*$ there exists an entire map $f_t \in \mathcal{E}^{p,q}(\theta)$ with the following properties:*

(i) *f_t is conjugate to f by a quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ which satisfies $\varphi = \zeta_{f_t} \circ \zeta_f^{-1}$ on $\gamma_{f,r}$ and $\overline{\partial}\varphi = 0$ off $\Omega_{f,r}$ for some $0 < r < 1$.*

(ii) *The free capture spot $\omega_t = \varphi(\omega) \in \Delta_{f_t} \smallsetminus \{0\}$ has conformal position t.*

The map f_t with these properties is unique. Moreover, the family $\{f_t\}_{t\in\mathbb{D}^*}$ depends holomorphically on t.

Proof. To show existence of f_t, let $t_0 := \zeta_f^{-1}(\omega) \in \mathbb{D}^*$ and fix a small ε and a radius r so that
$$0 < \varepsilon < \min\{|t|, |t_0|\} \leq \max\{|t|, |t_0|\} < r < 1.$$
Let $\beta : \mathbb{D} \to \mathbb{D}$ be a quasiconformal map such that
$$\beta(t_0) = t,$$
$$\beta \circ R_\theta = R_\theta \circ \beta, \tag{7.1}$$
and
$$\beta = \mathrm{id} \quad \text{in } \mathbb{D}_\varepsilon \cup \mathbb{A}_{r,1}. \tag{7.2}$$
The conformal structure $\mu := \beta^*(\mu_0)$ on $\mathbb{D}$ is R_θ-invariant and has bounded dilatation. Define a conformal structure ν on $\mathbb{C}$ by first setting $\nu := (\zeta_f^{-1})^*(\mu)$ on Δ_f, then spreading it along the iterated preimages of Δ_f using appropriate branches of f^{-n} and letting $\nu = \mu_0$ elsewhere. Evidently, ν is f-invariant and of bounded dilatation,

and $\nu = \mu_0$ off the iterated preimages of $\Delta_r \smallsetminus \overline{\Delta_\varepsilon}$. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal solution of $\varphi^*(\mu_0) = \nu$ normalized by the conditions $\varphi(0) = 0$ and $\varphi'(0) = 1$. The conjugate map $f_t := \varphi \circ f \circ \varphi^{-1}$ is holomorphic, hence it belongs to $\mathcal{E}^{p,q}$ by Corollary 3.2. Moreover, f_t has a Siegel disk $\Delta_{f_t} = \varphi(\Delta_f)$ of rotation number θ centered at 0. The composition $\varphi \circ \zeta_f \circ \beta^{-1} : \mathbb{D} \to \Delta_{f_t}$ preserves μ_0, conjugates R_θ to f_t, and has derivative 1 at the origin. Hence $\varphi \circ \zeta_f \circ \beta^{-1} = \zeta_{f_t}$ and the conformal radius of Δ_{f_t} is 1. Thus, $f_t \in \mathcal{E}^{p,q}(\theta)$ and

$$\zeta_{f_t}^{-1}(\omega_t) = (\beta \circ \zeta_f^{-1})(\omega) = \beta(t_0) = t,$$

which means the conformal position of ω_t is t. Uniqueness of f_t follows from Theorem 6.1.

It remains to show that f_t depends holomorphically on t. Fix $t_1 \in \mathbb{D}^*$, suppose $t_1 \neq t_0$, and construct the maps β, φ and the conformal structures μ, ν as above. Consider the conformal structure $s\mu$ on $\mathbb{D}$ for $|s| < 1+\delta$, where $\delta > 0$ is small enough to guarantee $s\mu$ has bounded dilatation. Let $\beta_s : \mathbb{D} \to \mathbb{D}$ be the unique solution of the Beltrami equation $\beta_s^*(\mu_0) = s\mu$ subject to the normalization $\beta_s(0) = 0$ and $\beta_s(1) = 1$. Then $\beta_0 = \mathrm{id}$, $\beta_1 = \beta$, and β_s depends holomorphically on s by the Measurable Riemann Mapping Theorem. Observe that $R_\theta^*(s\mu) = s\mu$ since R_θ is holomorphic. Hence $\beta_s \circ R_\theta \circ \beta_s^{-1}$ is a conformal automorphism of the disk, which can only be the rotation R_θ itself. It follows that β_s commutes with R_θ, so

$$\beta_s(e^{2\pi i n\theta} z) = e^{2\pi i n\theta} \beta_s(z)$$

for all integers n. For each $z \neq 0$ choose a sequence $\{n_k\}$ of integers such that $e^{2\pi i n_k \theta} \to |z|/z$ as $k \to \infty$. Substituting $n = n_k$ in the above equation and letting $k \to \infty$, we obtain

$$\beta_s(|z|) = \frac{|z|}{z}\, \beta_s(z)$$

whenever $0 < |z| < 1$. In other words, $z \mapsto \beta_s(z)/z$ depends only on $|z|$. Since this function is holomorphic in $\mathbb{D}_\varepsilon^* \cup \mathbb{A}_{r,1}$, it must be constant in each of $\mathbb{D}_\varepsilon^*$ and $\mathbb{A}_{r,1}$. The normalization $\beta_s(1) = 1$ gives $\beta_s(z) = z$ in $\mathbb{A}_{r,1}$, while we obtain $\beta_s(z) = a_s z$ in $\mathbb{D}_\varepsilon$, where $a_s \neq 0$ depends holomorphically on s.

Now let $\varphi_s : \mathbb{C} \to \mathbb{C}$ be the unique solution of $\varphi_s^*(\mu_0) = s\nu$ normalized so that $\varphi_s(0) = 0$ and $\varphi_s'(0) = a_s$. Then $\varphi_0 = \mathrm{id}$, $\varphi_1 = \varphi$, and φ_s also depends holomorphically on s. By a similar argument as above, the map $\varphi_s \circ f \circ \varphi_s^{-1}$ belongs to $\mathcal{E}^{p,q}(\theta)$ and its linearizing map is $\varphi_s \circ \zeta_f \circ \beta_s^{-1}$. It follows from the uniqueness of the family $\{f_t\}$ that

$$\varphi_s \circ f \circ \varphi_s^{-1} = f_{\beta_s(t_0)}. \tag{7.3}$$

The non-constant holomorphic function $s \mapsto t = \beta_s(t_0)$ sends a neighborhood of $s = 1$ onto a neighborhood of $\beta_1(t_0) = t_1$. Let $t \mapsto s(t)$ be a local inverse branch of this map defined on a small slit-disk neighborhood N of t_1. By (7.3), the map $t \mapsto f_t$ from N to $\mathcal{E}^{p,q}(\theta)$ can be written as a composition of holomorphic maps

$$t \mapsto s(t) \mapsto \varphi_{s(t)} \circ f \circ \varphi_{s(t)}^{-1},$$

so is itself holomorphic. Since this is true of every $t_1 \neq t_0$ and every choice of the small slit-disk neighborhood N of t_1, it follows that $t \mapsto f_t$ is holomorphic in $\mathbb{D}^*$. $\square$

For simplicity, we denote the Siegel disk Δ_{f_t} by Δ_t, the invariant curves $\gamma_{f_t,r}$ by $\gamma_{t,r}$, and the linearizing map ζ_{f_t} by ζ_t.

Lemma 7.2. *The family of linearizing maps $\zeta_t : \mathbb{D} \to \Delta_t$ depends holomorphically on $t \in \mathbb{D}^*$.*

Proof. Let $t_0 \in \mathbb{D}^*$. By Theorem 7.1, any two maps in the family $\{f_t\}$ are quasiconformally conjugate. The conjugacy maps repelling (resp. attracting) cycles to repelling (resp. attracting) cycles. It also maps indifferent cycles to indifferent cycles, preserving the multipliers. It follows that the repelling cycles of f_t move holomorphically without collision. Since these cycles are dense in the Julia set, the λ-lemma [**MSS**] implies there is a disk neighborhood N of t_0 over which $J(f_t)$ moves holomorphically. This holomorphic motion restricts to a motion of $\partial\Delta_t$ over N. As Sullivan shows in [**Su**], this implies the existence of a holomorphic family of Riemann maps $\{\chi_t : \mathbb{D} \to \Delta_t\}_{t\in N}$ with $\chi_t(0) = 0$. By Schwarz lemma, $\chi_t(z) = \zeta_t(\lambda_t z)$ for some constant λ_t with $|\lambda_t| = 1$. By the normalization $\zeta_t'(0) = 1$, we see that

$$\lambda_t = \chi_t'(0) = \frac{1}{2\pi i} \int_{|z|=1/2} \frac{\chi_t(z)}{z^2}$$

depends holomorphically on $t \in N$ as well, so $\lambda_t = \lambda$ is in fact independent of t. It follows that for each fixed z, the map $t \mapsto \zeta_t(z) = \chi_t(\lambda^{-1}z)$ is holomorphic in N. $\square$

Lemma 7.3. *For each $0 < r < 1$ there is a constant $K(r, f) > 1$ such that the invariant curve $\gamma_{t,r} \subset \Delta_t$ is a $K(r, f)$-quasicircle whenever $0 < |t| < 1/2$.*

The proof shows that the constant K actually depends on r and the family $\{f_t\}$ (and not its individual element f). The distinction is however immaterial since soon we will show that K can be chosen independent of both r and $\{f_t\}$ (see Lemma 7.6).

Proof. The family of linearizing maps $\{\zeta_t : \mathbb{D} \to \Delta_t\}_{t\in\mathbb{D}^*}$ is normal, so any limit function of ζ_t as $t \to 0$ is normalized and univalent in $\mathbb{D}$. By Lemma 7.2, for each $z \in \mathbb{D}$ the map $t \mapsto \zeta_t(z)$ is holomorphic in $\mathbb{D}^*$ and stays bounded as $t \to 0$ by normality. Hence $t = 0$ is a removable singularity of this map. Setting $\zeta_0(z) := \lim_{t\to 0} \zeta_t(z)$, it follows that the extended family $\{\zeta_t\}_{t\in\mathbb{D}}$ depends holomorphically on t, and $\zeta_t \to \zeta_0$ locally uniformly in $\mathbb{D}$ as $t \to 0$.

Now fix $0 < r < 1$. By Slodkowski's improved λ-lemma [**Sl**], the holomorphic motion

$$\zeta_t \circ \zeta_0^{-1} : \zeta_0(\mathbb{T}_r) \to \gamma_{t,r}$$

of the Jordan curve $\zeta_0(\mathbb{T}_r)$ extends to a holomorphic motion of the plane $\mathbb{C}$ which is $(1+|t|)/(1-|t|)$-quasiconformal. If $K(r,f)$ denotes the quasicircle constant of $\zeta_0(\mathbb{T}_r)$, it follows that $\gamma_{t,r}$ is a $3K(r,f)$-quasicircle whenever $0<|t|<1/2$. $\square$

7.2. **Surgery on the family $\{f_t\}$.** We now look at the effect of the surgery map $\mathcal{S}_r$ of §4 on the family $\{f_t\}$. Fix $0<r<1$. The quasiconformal map of Theorem 4.6, which initially conjugates f_t to $\mathcal{S}_r(f_t)$ off $\Delta_{t,r}$ only, can be easily modified, first inside $\Delta_{t,r}$ and then along $\Omega_{t,r}$ by pull-backs, to obtain a global quasiconformal conjugacy between f_t and $\mathcal{S}_r(f_t)$. It follows from the uniqueness part of Theorem 7.1 that $\mathcal{S}_r(f_t)$ must belong to the family $\{f_t\}$. Thus, at the level of parameters, $\mathcal{S}_r$ induces a map $\sigma_r:\mathbb{D}^*\to\mathbb{D}^*$ so that

$$\mathcal{S}_r(f_t)=f_{\sigma_r(t)}.$$

Lemma 7.4. *For each $0<r<1$,*

$$\sigma_r(t)=t \qquad \text{whenever } r<|t|<1.$$

Hence there is a constant K, depending only on p,q,θ, such that the invariant curve $\gamma_{t,r}$ is a K-quasicircle whenever $r<|t|<1$.

Proof. When $r<|t|<1$, every critical orbit of f_t hitting $\Delta_{t,r}$ must land at 0. In this case the assumptions of Theorem 6.1 hold for $f=f_t$, $g=\mathcal{S}_r(f_t)$ and the quasiconformal conjugacy $\varphi=\varphi_t$ given by Theorem 4.6. It follows that $\mathcal{S}_r(f_t)=f_t$, as required.

The second assertion follows from Corollary 5.8. $\square$

Lemma 7.5. *For each $0<r<1$,*

$$\lim_{t\to 0}\sigma_r(t)=0.$$

Proof. Recall the maps involved in the construction of $\mathcal{S}_r(f_t)$: the reflection $I_t:\widehat{\mathbb{C}}\to\widehat{\mathbb{C}}$ (§4.1), the standard extension $H_t:\mathbb{D}\to\mathbb{D}$ and the quasiconformal maps $\xi_t,\psi_t:\mathbb{C}\to\mathbb{C}$ (§4.4). Since by Corollary 4.7, $z\mapsto(\psi_t\circ H_t^{-1})(z/r)$ is the linearizing map for $f_{\sigma_r(t)}$, it is not hard to see that

$$\sigma_r(t)=rH_t(\xi_t(\omega_t)), \tag{7.4}$$

where $\omega_t=\zeta_t(t)$ is the free capture spot of f_t. By Lemma 7.3 and Corollary 4.1, I_t and hence ξ_t can be chosen $K(r,f)$-quasiconformal whenever $0<|t|<1/2$. The map $\xi_t\circ\zeta_t:\mathbb{D}_r\to\mathbb{D}$ is $K(r,f)$-quasiconformal fixing the origin, hence uniformly Hölder continuous of exponent $1/K(r,f)$. It follows that

$$\lim_{t\to 0}\xi_t(\omega_t)=\lim_{t\to 0}(\xi_t\circ\zeta_t)(t)=0.$$

By Corollary 5.7, the standard extension $H_t:\mathbb{D}\to\mathbb{D}$ is K-quasiconformal for some K independent of t and r. Since $H_t(0)=0$,

$$\lim_{t\to 0}H_t(\xi_t(\omega_t))=0.$$

This, in view of (7.4), proves the lemma. □

We can now prove the following improvement of Lemma 7.3:

Lemma 7.6. *There exists a constant K, depending only on p, q, θ, such that the invariant curve $\gamma_{t,r}$ is a K-quasicircle whenever $0 < r < 1$ and $0 < |t| < 1/2$.*

Proof. Let $0 < r < 1$ and K be the constant given by Corollary 5.8, so the invariant curve $\gamma_{\sigma_r(t),r}$ is a K-quasicircle for all $t \in \mathbb{D}^*$. Letting $t \to 0$ and making use of Lemma 7.5, we see by an argument similar to the proof of Theorem 2.3 that the Jordan curve $\zeta_0(\mathbb{T}_r)$ is a K-quasicircle. The last part of the proof of Lemma 7.3 then shows that $\gamma_{t,r}$ is a $3K$-quasicircle whenever $0 < |t| < 1/2$. □

Theorem 7.7. *There exists a constant K, depending only on p, q, θ, such that the invariant curve $\gamma_{t,r}$ is a K-quasicircle whenever $0 < r < 1$ and $t \in \mathbb{D}^*$.*

Proof. Fix $0 < r < 1$ and four distinct points $a, b, c, d \in \mathbb{T}$ (in this cyclic order). Define a holomorphic map $Z : \mathbb{D}^* \to \mathbb{C}$ by

$$Z(t) := \mathbf{Cr}(\zeta_t(ra), \zeta_t(rb), \zeta_t(rc), \zeta_t(rd)),$$

where $\mathbf{Cr}$ is the cross-ratio given by (2.3). By Theorem 2.5, Lemma 7.4 and Lemma 7.6, there exists a constant $M > 1$, depending only on p, q, θ, such that

$$|Z(t)| \leq M \qquad \text{if } r < |t| < 1 \text{ or } 0 < |t| < 1/2.$$

It follows from the Maximum Modulus Principle that $|Z| \leq M$ throughout $\mathbb{D}^*$. Since this holds for every $0 < r < 1$ and every quadruple (a, b, c, d), another application of Theorem 2.5 shows that $\gamma_{t,r}$ is a K-quasicircle for some $K > 1$ which depends only on M, hence only on p, q, θ. □

Proof of the Main Theorem when f has one free capture spot. Embed f in the family $\{f_t\}$ given by Theorem 7.1. By Theorem 7.7 the invariant curve $\gamma_{f,r}$ is a K-quasicircle for some K depending only on p, q, θ, hence independent of r. The rest of the argument is as before. □

8. The general case

Now we address the case of an $f \in \mathcal{E}^{p,q}(\theta)$ with two or more free capture spots. We will perform a cut-and-paste surgery on f to construct a new map $g \in \mathcal{E}^{p,q}(\theta)$ with at most one free capture spot. Even though g is no longer conjugate to f, there is a quasiconformal map of the plane which sends $\partial\Delta_f$ to $\partial\Delta_g$. The special cases of the Main Theorem proved in §6 and §7 then show that $\partial\Delta_g$ is a quasicircle passing through a critical point. Hence the same must be true of $\partial\Delta_f$.

8.1. **The preimages of Δ_r.** Fix an $f \in \mathcal{E}^{p,q}(\theta)$. For simplicity, we once again drop the subscript f from our notations. According to [**EL**], all the Fatou components of a transcendental entire map with a bounded set of critical and asymptotic values must be simply-connected. In particular, the connected components of $f^{-n}(\Delta)$ are simply-connected Fatou components, which can be bounded or unbounded (in the non-polynomial case where $q > 0$, unbounded preimages of Δ always exist). For each $n \geq 1$, set

$$\Gamma_n := \text{ the collection of the connected components of } f^{-n}(\Delta) \smallsetminus f^{-n+1}(\Delta).$$

If $U \in \Gamma_n$ for some $n > 1$, Corollary 3.4 shows that $f(U) \in \Gamma_{n-1}$. However, if $U \in \Gamma_1$, the image $f(U)$ is either Δ or $\Delta \smallsetminus \{0\}$.

The *capture radius* of f is the number in $[0, 1)$ defined by

$$\kappa := \max\{|\zeta^{-1}(\hat{c})| : \hat{c} \text{ is a capture spot of } f\},$$

where $\zeta : \mathbb{D} \to \Delta$ is the linearizing map for f. Alternatively, κ is the smallest radius r for which the annulus $\Delta \smallsetminus \overline{\Delta_r}$ is disjoint from the critical orbits of f. Note that $\kappa = 0$ if and only if f has no free capture spot.

Let $0 < r < 1$ and $n \geq 1$. For each $U \in \Gamma_n$, define

$$U_r := f^{-n}(\Delta_r) \cap U.$$

Lemma 8.1. *U_r is a simply-connected domain whenever $\kappa < r < 1$.*

The proof shows that for every $0 < r < 1$, each component of U_r is simply-connected. We need the condition $\kappa < r < 1$ only to guarantee U_r is connected.

Proof. Let W be a component of U_r, η be a simple closed curve in W and O be the bounded component of $\mathbb{C} \smallsetminus \eta$. Since $\partial O = \eta \subset W$, we have $f^{\circ n}(z) \in \Delta_r$ if $z \in \partial O$. Since $f^{\circ n}$ is an open mapping, the same holds if $z \in O$, proving $O \subset W$. Thus, W is simply-connected.

To show connectivity of U_r, take any s with $\kappa < s < r$ and note that by Theorem 3.3 the restriction of $f^{\circ n}$ to each component of $U \smallsetminus \overline{U_s}$ is a covering map onto $\Delta \smallsetminus \overline{\Delta_s}$. Hence the radial foliation of $\Delta \smallsetminus \overline{\Delta_s}$ pulls back under f^{-n} to a foliation of $U \smallsetminus \overline{U_s}$ by analytic arcs. Moreover, each leaf of the latter foliation lands at a well-defined point of ∂U_s. Hence we can define a deformation retraction $U \to \overline{U_s}$ by sending each point of $U \smallsetminus \overline{U_s}$ to the landing point of the leaf passing through it, and keeping $\overline{U_s}$ fixed pointwise. Thus $\overline{U_s}$ is connected and by letting $s \to r$ it follows that U_r must be connected also. □

8.2. **Action of f on immediate preimages of Δ_r.** Throughout we fix a radius r such that $\kappa < r < 1$.

Lemma 8.2. *Suppose $U \in \Gamma_1$ and U_r is bounded. Then ∂U_r is a Jordan curve and $f : U_r \to \Delta_r$ is a finite-degree branched covering.*

Proof. By Lemma 3.8, U_r is bounded by a Jordan curve. Evidently $f(U_r) \subset \Delta_r$ and $f(\partial U_r) \subset \partial \Delta_r = \gamma_r$. Hence $f : U_r \to \Delta_r$ is a proper holomorphic map. As such, it must be a finite-degree branched covering. $\square$

Here is the corresponding statement when U_r is unbounded:

Lemma 8.3. *Suppose $U \in \Gamma_1$ and U_r is unbounded. Then ∂U_r is a disjoint union of finitely many simple analytic arcs tending to ∞ in both directions. The map $f : U_r \to \Delta_r$ or $\Delta_r \smallsetminus \{0\}$ is an infinite-degree branched covering with finitely many branched points. More precisely, if $\varphi : U_r \to \mathbb{D}$ and $\psi : \Delta_r \to \mathbb{D}$ are conformal isomorphisms with $\psi(0) = 0$, then the induced map $F := \psi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ is an inner function of the form*

$$F(z) = B(z) \exp\left(\frac{A(z)+1}{A(z)-1}\right); \tag{8.1}$$

here $A, B : \mathbb{D} \to \mathbb{D}$ are finite Blaschke products, the degree of A is equal to the number of ends of U_r (equivalently, the number of components of ∂U_r), and the degree of B is equal to the number of zeros of f in U_r.

Proof. The claim on ∂U_r follows from Lemma 3.8 since $\partial U_r \subset f^{-1}(\gamma_r)$ and $f^{-1}(\gamma_r)$ has finitely many components. None of these components is a Jordan curve since U_r is simply-connected and unbounded.

Let $B : \mathbb{D} \to \mathbb{D}$ be a finite Blaschke product having the same zeros of the same multiplicities as F (set $B \equiv 1$ if F has no zeros). The map $G := F/B : \mathbb{D} \to \mathbb{D}^*$ lifts under the universal covering $\mathbb{D} \to \mathbb{D}^*$ given by $z \mapsto \exp((z+1)/(z-1))$ to a holomorphic map $A : \mathbb{D} \to \mathbb{D}$. We show that A is proper and its degree is equal to the number of ends of U_r.

The conformal isomorphisms φ and ψ extend analytically across the boundaries ∂U_r and $\gamma_r = \partial \Delta_r$. The image $\varphi(\partial U_r)$ is of the form $\mathbb{T} \smallsetminus \{a_1, \ldots, a_k\}$, where $a_1, \ldots, a_k$ are distinct and k is the number of ends of U_r. Since $f(\partial U_r) = \gamma_r$, it follows that F (hence G) extends analytically across each of the k arcs of $\mathbb{T} \smallsetminus \{a_1, \ldots, a_k\}$, mapping these arcs to $\mathbb{T}$ (however, the representation (8.1) will show that the non-tangential limit of F (hence G) at each a_j is 0).

Lifting under $z \mapsto \exp((z+1)/(z-1))$, we see that A extends analytically across each arc of $\mathbb{T} \smallsetminus \{a_1, \ldots, a_k\}$, mapping these arcs to $\mathbb{T}$. To check properness of A, it is therefore enough to show that every sequence $\{z_i\}$ in $\mathbb{D}$ converging to some a_j has a subsequence for which $A(z_i) \to 1$. This is evident if there is a subsequence of $\{z_i\}$ for which $F(z_i) \to 0$. Otherwise $\{F(z_i)\}$ stays away from 0. After passing to a subsequence, we may assume that $\{F(z_i)\}$ is contained in an annulus $\mathbb{A}_{a,b}$ $(0 < a < b \leq 1)$ containing no critical value of F (if all but finitely many of the $F(z_i)$ happen to be on a circle $\mathbb{T}_s$ which contains a critical value of F, simply replace each z_i by a generic z_i' nearby so that $|F(z_i')| < s$ and $|A(z_i) - A(z_i')| < 1/i$). By Lemma 3.8, the preimage of each boundary circle of $\mathbb{A}_{a,b}$ has finitely many components, so $F^{-1}(\mathbb{A}_{a,b})$ must have finitely many components as well. Moreover, each such component is either an annulus compactly contained in $\mathbb{D}$ or else simply-connected having accumulation points on $\mathbb{T}$. After passing to a further subsequence, we may assume that $\{z_i\}$ is contained a single component W of $F^{-1}(\mathbb{A}_{a,b})$, which must be simply-connected since $\{z_i\} \subset W$ tends to $a_j \in \mathbb{T}$ so W cannot be compactly contained in $\mathbb{D}$. Thus, the map $F : W \to \mathbb{A}_{a,b}$ is a universal covering and there is a conformal isomorphism $\hat{F} : W \to \{z : \log a < \operatorname{Re}(z) < \log b\}$ such that $F = \exp(\hat{F})$. Since $\{z_i\}$ tends to a_j, it follows that

$$|\operatorname{Im}(\hat{F}(z_i))| \to \infty \quad \text{as } i \to \infty. \tag{8.2}$$

On the other hand, W is simply connected and B has no zeros in there, so there is a lift $\hat{B} : W \to \mathbb{C}$ satisfying $B = \exp(\hat{B})$. Moreover, since $\{B(z_i)\}$ converges to $B(a_j) \in \mathbb{T}$, the sequence $\{\hat{B}(z_i)\}$ has to tend to a well-defined value of $\log B(a_j)$ on the imaginary axis. In particular,

$$|\operatorname{Im}(\hat{B}(z_i))| \ \text{ remains bounded as } i \to \infty. \tag{8.3}$$

Now both $(A+1)/(A-1)$ and $\hat{F} - \hat{B}$ are lifts of G in W under the exponential map, so after adding an appropriate integer multiple of $2\pi i$ to $\hat{B}$ we can arrange that

$$\frac{A+1}{A-1} = \hat{F} - \hat{B} \quad \text{throughout } W.$$

By (8.2) and (8.3), the imaginary part of the left side along $\{z_i\}$ is unbounded. This easily implies $A(z_i) \to 1$, as required.

Thus, as a proper holomorphic map, $A : \mathbb{D} \to \mathbb{D}$ is a finite Blaschke product. The above argument shows that $A^{-1}(1) = \{a_1, \dots, a_k\}$, so the degree of A is k. □

Corollary 8.4. *Suppose $U \in \Gamma_1$ and U_r is unbounded. Let*

$$\begin{aligned} k &:= \textit{the number of ends of } U_r, \\ \ell &:= \textit{the number of zeros of } f \textit{ in } U_r, \\ m &:= \textit{the number of critical points of } f \textit{ in } U_r. \end{aligned}$$

Then

$$m = k + \ell - 1.$$

Proof. The Blaschke products A and B of Lemma 8.3 have degrees k and ℓ, respectively. The critical points of F are the roots of the rational equation

$$B'(z) - \frac{2A'(z)B(z)}{(A(z)-1)^2} = 0,$$

so there are $2(k + \ell - 1)$ of them counting multiplicities. Since F commutes with the reflection $z \mapsto 1/\overline{z}$ and has no critical point on $\mathbb{T}$, precisely half of its critical points must be in $\mathbb{D}$, which shows $m = k + \ell - 1$. □

8.3. **Action of f on further preimages of Δ_r.** We continue assuming $\kappa < r < 1$.

Lemma 8.5. *Suppose $U \in \Gamma_n$ for some $n > 1$, so $V := f(U) \in \Gamma_{n-1}$. Then U_r is bounded if and only if V_r is bounded. The map $f : U_r \to V_r$ is always a finite-degree branched covering.*

Proof. Suppose first that U_r is bounded. Then clearly $V_r = f(U_r)$ is also bounded and ∂U_r and ∂V_r are both analytic Jordan curves. The inclusion $f(\partial U_r) \subset \partial V_r$ shows that $f : U_r \to V_r$ is proper, hence a finite-degree branched covering.

Now suppose U_r is unbounded. We have $V \neq \Delta$ since $n > 1$. By Lemma 3.7, $f : U_r \to V_r$ is a proper map, hence a finite-degree branched covering. To see V_r is also unbounded in this case, suppose by way of contradiction that V_r is bounded by a Jordan curve. Choose conformal isomorphisms $\varphi : U_r \to \mathbb{D}$ and $\psi : V_r \to \mathbb{D}$ so the map $B := \psi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ is proper. Take an $a \in \mathbb{T}$ such that the radial line L landing at a has its preimage $\eta := \varphi^{-1}(L) \subset U_r$ tending to ∞. The image $f(\eta) \subset V_r$ lands at the well-defined point $\psi^{-1}(B(a)) \in \partial V_r$. But this would contradict Theorem 3.5. □

Remark 8.6. We can now conclude that for each $U \in \Gamma_n$, the boundary ∂U_r has finitely many components. For $n = 1$, this follows from Lemma 3.8, and the general case follows from Lemma 8.5 by induction on n.

8.4. **Modifying f on critical preimages of Δ_r.** Suppose that $f \in \mathcal{E}^{p,q}(\theta)$ has two or more free capture spots and $\max\{\kappa, 1/2\} < r < 1$. We first assign a *center* o_U to each iterated preimage U of the Siegel disk Δ as follows. Set $o_\Delta := 0$ and $\omega := \zeta(1/2) \in \Delta_r$. If $U \in \Gamma_1$ and U_r is bounded, then $f(U_r) = \Delta_r$ (Lemma 8.2) and we choose the center o_U arbitrarily from the finite set $f^{-1}(0) \cap U$. If $U \in \Gamma_1$ and U_r is unbounded, then $f(U_r) \supset \Delta_r \smallsetminus \{0\}$ (Lemma 8.3) and we choose the center o_U from the infinite set $f^{-1}(\omega) \cap U$. Suppose $n > 1$ and we have defined the centers of all the iterated preimages of Δ in Γ_j for $1 \leq j \leq n-1$. If $U \in \Gamma_n$, then $V := f(U) \in \Gamma_{n-1}$ and $f(U_r) = V_r$ (Lemma 8.5), and we choose the center o_U from the finite set $f^{-1}(o_V) \cap U$. This finishes the inductive definition of the centers. Note that the assignment $U \mapsto o_U$ respects the action of f:

$$f(o_U) = o_{f(U)} \qquad \text{for all } U \in \bigcup_{n \geq 2} \Gamma_n.$$

Now suppose $U \in \Gamma_n$ for some $n \geq 1$ and U_r contains at least one critical point of f. We will modify f on U_r so that the new quasiregular map has a single branch point at o_U. We will distinguish three cases:

- *Case 1.* U_r is bounded. If $V := f(U)$, it follows from Lemma 8.5 that V_r is also bounded, both ∂U_r and ∂V_r are analytic Jordan curves and $f : U_r \to V_r$ is a branched covering of some degree $d \geq 2$. Take quasiconformal maps $\varphi : U_r \to \mathbb{D}$ and

$\psi : V_r \to \mathbb{D}$, with $\varphi(o_U) = \psi(o_V) = 0$, such that

$$f = \psi^{-1} \circ \pi_d \circ \varphi \quad \text{on} \quad \partial U_r,$$

where $\pi_d : z \mapsto z^d$. Replace f in U_r with

$$\hat{f} := \psi^{-1} \circ \pi_d \circ \varphi.$$

Thus $\hat{f} = f$ on ∂U_r and $\hat{f} : U_r \to V_r$ is a degree d quasiregular branched covering with a single branch point at o_U which is ramified over o_V.

• *Case 2.* U_r is unbounded and $n > 1$. If $V := f(U)$, it follows from Lemma 8.5 that V_r is unbounded and $f : U_r \to V_r$ is a branched covering of some degree $d \geq 2$. Take conformal isomorphisms $\varphi : U_r \to \mathbb{D}$ and $\psi : V_r \to \mathbb{D}$, with $\varphi(o_U) = \psi(o_V) = 0$. The induced map $B := \psi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ is proper, hence a degree d Blaschke product. If $0 < s < 1$ is close to 1, there are quasiconformal maps $\hat{\varphi} : B^{-1}(\mathbb{D}_s) \to \mathbb{D}$ and $\hat{\psi} : \mathbb{D}_s \to \mathbb{D}$, both fixing the origin, such that

$$\hat{\psi} \circ B \circ \hat{\varphi}^{-1} = \pi_d \quad \text{on} \quad \mathbb{T}.$$

Replace f in U_r with

$$\hat{f} := \begin{cases} \psi^{-1} \circ \hat{\psi}^{-1} \circ \pi_d \circ \hat{\varphi} \circ \varphi & \text{in } \varphi^{-1}(B^{-1}(\mathbb{D}_s)) \\ f & \text{elsewhere in } U_r. \end{cases}$$

Note that the simply-connected domain $\varphi^{-1}(B^{-1}(\mathbb{D}_s))$ is compactly contained in U_r, so this modification does not change f near ∞. As in *Case 1*, the map $\hat{f} : U_r \to V_r$ is a degree d quasiregular branched covering with a single branch point at o_U which is ramified over o_V.

• *Case 3.* U_r is unbounded and $n = 1$. Take a conformal isomorphism $\varphi : U_r \to \mathbb{D}$ with $\varphi(o_U) = 0$ and let $\psi := (1/r)\zeta^{-1}$ where $\zeta : \mathbb{D} \to \Delta$ is the linearizing map of f. By Lemma 8.3, the induced map $F := \psi \circ f \circ \varphi^{-1} : \mathbb{D} \to \mathbb{D}$ has the form (8.1), where $A, B : \mathbb{D} \to \mathbb{D}$ are finite Blaschke products, $\deg A = k \geq 1$ is the number of ends of U_r and $\deg B = \ell \geq 0$ is the number of zeros of f in U_r.

To modify f in U_r in a way similar to the *Case 1* or *2* above, we will need the following

Lemma 8.7. *There exists an infinite-degree quasiregular branched covering map $\hat{F}$: $\mathbb{D} \to \mathbb{D}$ with ℓ zeros and a single branch point of local degree $k + \ell$ at the origin ramified over $\hat{F}(0) = 1/2$, which coincides with F in a neighborhood of $\mathbb{T}$.*

Proof. We first construct an infinite-degree branched covering $G : \mathbb{D} \to \mathbb{D}$ with the required mapping properties. We then modify it to obtain a quasiregular map $\hat{F}$ which has the additional property that it coincides with F near $\mathbb{T}$. For $1 \leq j \leq k$, let $b_j := e^{2\pi i j/k}$. Take "petals" $\{\Pi_j\}_{1 \leq j \leq k+\ell}$ in $\mathbb{D}$ such that (i) $\overline{\Pi_j} \cap \overline{\Pi_k} = \{0\}$ for $j \neq k$;

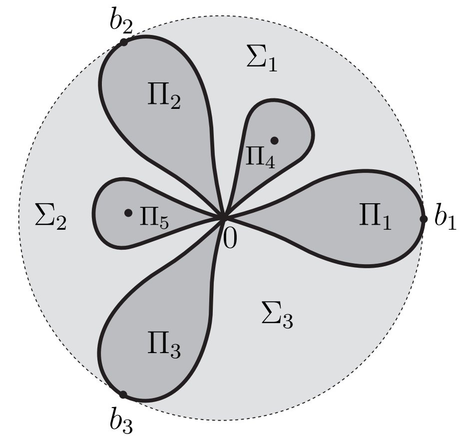

FIGURE 6. *Petals Π_j and strips Σ_j used in the construction of G. Here $k=3$ and $\ell=2$. The dots in Π_4 and Π_5 indicate the zeros of G.*

(ii) $\partial\Pi_j$ is smooth except at 0; (iii) for $1 \leq j \leq k$, $\partial\Pi_j$ is tangent to $\mathbb{T}$ at b_j; (iv) for $k+1 \leq j \leq k+\ell$, Π_j is compactly contained in $\mathbb{D}$. For each $1 \leq j \leq k$, define $G : \Pi_j \to \mathbb{D}^*_{1/2}$ to be a universal covering so that $G(0) = 1/2$ and $G(z) \to 0$ as $z \to b_j$ non-tangentially. Similarly, for each $k+1 \leq j \leq k+\ell$, define $G : \Pi_j \to \mathbb{D}_{1/2}$ to be a conformal isomorphism so that $G(0) = 1/2$. The complement $\mathbb{D} \smallsetminus \bigcup_{j=1}^{k+\ell} \overline{\Pi_j}$ is the union of strips $\{\Sigma_j\}_{1\leq j\leq k}$, where the labeling is chosen so that Σ_j has ends at b_j and b_{j+1} mod k (see Fig. 6). Define $G : \Sigma_j \to \mathbb{A}_{1/2,1}$ to be a universal covering compatible on the boundary with the definition of G on the petals. Evidently $G : \mathbb{D} \to \mathbb{D}$ can be constructed in a piecewise smooth fashion. The resulting map is an infinite-degree branched covering with a unique branch point of local degree $k+\ell$ at 0 ramified over $G(0) = 1/2$. Moreover, G has ℓ zeros, one in each petal Π_j for $k+1 \leq j \leq k+\ell$.

With $F = B \exp((A+1)/(A-1))$ as above, let $A^{-1}(1) := \{a_1, \ldots, a_k\}$, where we label the a_j counterclockwise. For small enough $\varepsilon > 0$, there is a component D_j of $F^{-1}(\mathbb{D}_\varepsilon)$ homeomorphic to a disk which is tangent to $\mathbb{T}$ at a_j, and the map $F : D_j \to \mathbb{D}^*_\varepsilon$ is covering. Similarly, there is a component S_j of $F^{-1}(\mathbb{A}_{1-\varepsilon,1})$ homeomorphic to a strip which has ends at a_j and a_{j+1} mod k, and the map $F : S_j \to \mathbb{A}_{1-\varepsilon,1}$ is covering. Take a radial segment L from the circle $\mathbb{T}_\varepsilon$ to the circle $\mathbb{T}_{1-\varepsilon}$ which avoids the critical values of F. For each j take a lift R_j of L which connects ∂S_j to ∂D_j near b_j and another lift L_{j+1} of L which connects ∂S_j to ∂D_{j+1} near b_{j+1} mod k (note that there are countably many choices for such lifts). Let I_j be the segment on ∂S_j from R_j to L_{j+1} and J_j be the segment on ∂D_j from L_j to R_j. Let Θ be the positively oriented

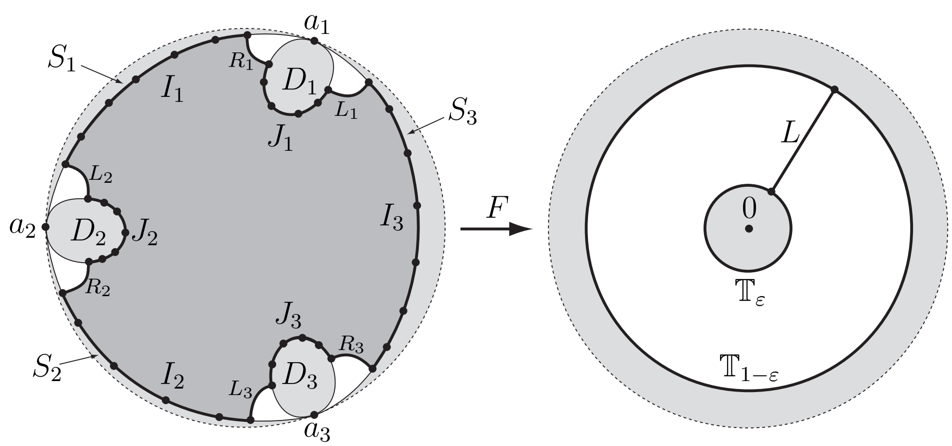

FIGURE 7. *Construction of the Jordan curve* Θ *for the map* F. *Here* $k = 3$ *and* $\ell = 2$. *The curves* $F(I_1), F(I_2), F(I_3)$ *have winding numbers* $6, 5, 7$ *and the curves* $F(J_1), F(J_2), F(J_3)$ *have winding numbers* $-5, -6, -5$, *respectively. Hence* $F(\Theta)$ *has winding number* 2, *which is equal to* ℓ *as in* (8.4).

Jordan curve in $\mathbb{D}$ formed by concatenating the following arcs:

$$L_1 \to J_1 \to R_1 \to I_1 \to \cdots \to L_k \to J_k \to R_k \to I_k.$$

By the Argument Principle,

$$\ell = \operatorname{ind}(F(\Theta), 0) = \sum_{j=1}^{k} \operatorname{ind}(F(I_j), 0) - \sum_{j=1}^{k} \operatorname{ind}(F(J_j), 0), \tag{8.4}$$

where $\operatorname{ind}(\cdot, 0)$ denotes the winding number (compare Fig. 7).

There is a completely similar construction for the map G, namely, let $D'_j := G^{-1}(\mathbb{D}_\varepsilon) \cap \Pi_j$ and $S'_j := G^{-1}(\mathbb{A}_{1-\varepsilon,1}) \cap \Sigma_j$, choose arcs R'_j, L'_j, I'_j, J'_j and construct the Jordan curve Θ' by the corresponding concatenation of these arcs. The point is that for any choice of the lifts R'_j and L'_j, we also have

$$\ell = \operatorname{ind}(G(\Theta'), 0) = \sum_{j=1}^{k} \operatorname{ind}(G(I'_j), 0) - \sum_{j=1}^{k} \operatorname{ind}(G(J'_j), 0),$$

so by selecting the lifts R_j, L_j, R'_j, L'_j sufficiently close to the corresponding ends we can arrange $\operatorname{ind}(G(I'_j), 0) = \operatorname{ind}(F(I_j), 0)$ and $\operatorname{ind}(G(J'_j), 0) = \operatorname{ind}(F(J_j), 0)$ for all j.

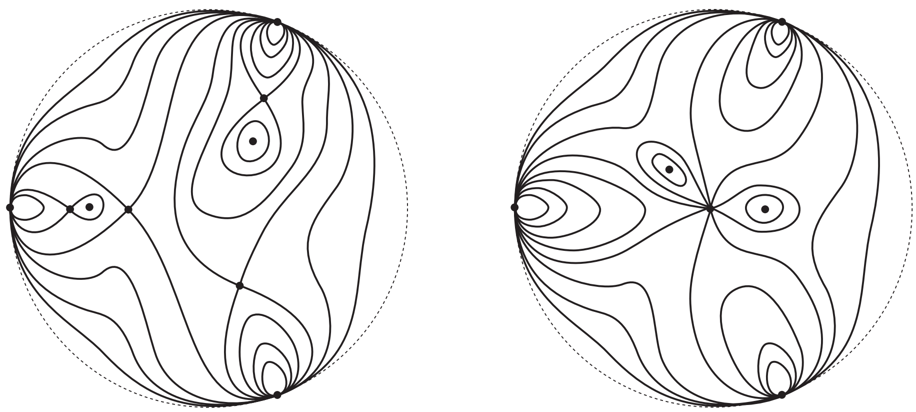

FIGURE 8. *Left: The singular foliation* $|F| = \text{const.}$, *where* $F : \mathbb{D} \to \mathbb{D}$ *is the inner function induced by* $f : U_r \to \Delta_r$ *when* U_r *is unbounded. In this example,* F *has* $m = 4$ *simple critical points and* $\ell = 2$ *zeros, and the* $k = 3$ *marked points on* $\partial\mathbb{D}$ *correspond to the ends of* U_r *near infinity, so* $k + \ell - m = 1$ *as in Corollary 8.4. Right: The singular foliation* $|\hat{F}| = \text{const.}$ *of the modified quasiregular map* $\hat{F} : \mathbb{D} \to \mathbb{D}$ *of Lemma 8.7, with a single branch point of local degree* $k+\ell = m+1 = 5$ *at the origin.*

Once this is achieved, we can define a piecewise smooth homeomorphism $\Phi : \Theta \to \Theta'$ which satisfies $F = G \circ \Phi$, simply by following F with an appropriate branch of G^{-1}. As Θ and Θ' are quasicircles (their finitely many corners have all $90°$ angles), there is a quasiconformal extension of Φ mapping the interior of Θ to that of Θ', with $\Phi(0) = 0$. The map $\hat{F} : \mathbb{D} \to \mathbb{D}$ defined by

$$\hat{F} := \begin{cases} G \circ \Phi & \text{inside } \Theta \\ F & \text{elsewhere} \end{cases}$$

has the required properties. $\square$

Fig. 8 illustrates an example.

Back to our discussion of modifying f, we can now replace f in U_r with

$$\hat{f} := \psi^{-1} \circ \hat{F} \circ \varphi$$

The map $\hat{f} : U_r \to \Delta_r$ is an infinite-degree quasiregular branched covering with a single branch point of local degree $k + \ell$ at o_U which is ramified over $\omega := \zeta(r/2)$. Note that this modification does not change f near ∞.

Proof of the Main Theorem in the general case. Let $f \in \mathcal{E}^{p,q}(\theta)$ have two or more free capture spots. Define the modified map $\hat{f}$ on every iterated preimage of Δ_f containing a critical point, as above. Extend $\hat{f}$ to a quasiregular map $\mathbb{C} \to \mathbb{C}$ by setting it equal to f elsewhere. Note in particular that $\hat{f} = f$ in a neighborhood of ∞. The map $\hat{f}$ admits an invariant conformal structure μ of bounded dilatation: it suffices to set $\mu = \mu_0$ on Δ_f, define

$$\mu := (\hat{f}^{\circ n})^*(\mu) \qquad \text{on } \hat{f}^{-n}(\Delta_f) = f^{-n}(\Delta_f)$$

for every $n \geq 1$, and set $\mu = \mu_0$ elsewhere. This μ is clearly $\hat{f}$-invariant by the construction. It has bounded dilatation since $\hat{f}$ fails to be holomorphic on at most finitely many of the iterated preimages of Δ_f. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal map which solves the Beltrami equation $\varphi^*(\mu_0) = \mu$ and satisfies $\varphi(0) = 0, \varphi'(0) = 1$. The conjugate map $g := \varphi \circ \hat{f} \circ \varphi^{-1}$ preserves μ_0, hence is entire. Since g and $\hat{f}$ are quasiconformally conjugate and $\hat{f} = f$ near infinity, the proof of Corollary 3.2 shows that $g \in \mathcal{E}^{p,q}$. Since g has a Siegel disk of rotation number θ and conformal radius 1 centered at 0, we actually have $g \in \mathcal{E}^{p,q}(\theta)$. By the construction of $\hat{f}$, the map g has at most one free capture spot. It follows from the special cases of the Main Theorem proved in §6 and §7 that $\partial\Delta_g$ is a quasicircle passing through a critical point. Since $\partial\Delta_f = \varphi^{-1}(\partial\Delta_g)$, the same must be true of $\partial\Delta_f$, which completes the proof. □

References

[**A**] L. Ahlfors, *Lectures on quasiconformal mappings*, 2nd ed. with supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard, American Mathematical Society, Providence, RI, 2006.

[**B**] L. Bers, *On moduli of Kleinian groups*, Russ. Math. Surv. **29** (1974) 88-102.

[**C**] A. Chéritat, *Ghys-like models for Lavaurs and simple entire maps*, Conform. Geom. Dyn. **10** (2006) 227-256.

[**D**] A. Douady, *Disques de Siegel et anneaux de Herman*, Astérisque **152-153** (1987) 151-172.

[**DH**] A. Douady and J. Hubbard, *On the dynamics of polynomial-like maps*, Ann. Sci. École Norm. Sup. **2** (1985) 287-343.

[**EL**] A. Eremenko and M. Lyubich, *Dynamical properties of some classes of entire functions*, Ann. Inst. Fourier (Grenoble) **42** (1992) 989-1020.

[**Ge**] L. Geyer, *Siegel discs, Herman rings and the Arnold family*, Trans. Amer. Math. Soc. **353** (2001) 3661-3683.

[**Gh**] E. Ghys, *Transformations holomorphes au voisinage d'une courbe de Jordan*, C. R. Acad. Sci. Paris Sér. I Math. **298** (1984) 385-388.

[**GS**] J. Graczyk and G. Swiatek, *Siegel disks with critical points in their boundaries*, Duke Math. J. **119** (2003) 189-196.

[**H1**] M. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Inst. Hautes Études Sci. Publ. Math. **49** (1979) 5-233.

[**H2**] M. Herman, *Are there critical points on the boundaries of singular domains?*, Comm. Math. Phys. **99** (1985) 593-612.

[**H3**] M. Herman, *Uniformité de la distortion de Swiatek pour les familles compactes de produits de Blaschke*, manuscript, 1987 (English translation available at A. Chéritat's webpage `http://picard.ups-tlse.fr/~cheritat`)

[**KZ**] L. Keen and G. Zhang, *Bounded type Siegel disks of a one-dimensional family of entire functions*, preprint, 2007.

[**LV**] O. Lehto and K. Virtanen, *Quasiconformal mappings in the plane*, 2nd ed., Springer-Verlag, New York-Heidelberg, 1973.

[**MSS**] R. Mane, P. Sad, and D. Sullivan, *On the dynamics of rational maps*, Ann. Sci. École Norm. Sup. (4) **16** (1983) 193-217.

[**MNTU**] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, *Holomorphic dynamics*, Cambridge University Press, Cambridge, 2000.

[**P**] R. Perez-Marco, *Fixed points and circle maps*, Acta Math. **179** (1997) 243-294.

[**S**] M. Shishikura, *Herman's theorem on quasisymmetric linearization of analytic circle homeorphisms*, manuscript, 1990.

[**Si**] C. L. Siegel, *Iteration of analytic functions*, Ann. of Math. (2) **43** (1942) 607-612.

[**Sl**] Z. Slodkowski, *Holomorphic motions and polynomial hulls*, Proc. Amer. Math. Soc. **111** (1991) 347-355.

[**Su**] D. Sullivan, *Quasiconformal homeomorphisms and dynamics III: Topological conjugacy classes of analytic endomorphisms*, manuscript.

[**Y**] J.C. Yoccoz, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne*, Ann. Sci. École Norm. Sup. (4) **17** (1984) 333-359.

[**Z1**] S. Zakeri, *Dynamics of cubic Siegel polynomials*, Commun. Math. Phys. **206** (1999) 185-233.

[**Z2**] S. Zakeri, *Biaccessibility in quadratic Julia sets*, Ergodic Theory Dynam. Systems **20** (2000) 1859-1883.

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