# ON ARITHMETICAL DIFFERENCE OF TWO CANTOR SETS

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Abstract. We construct a large class of dynamically defined Cantor sets on the real line whose self-difference sets are Cantor sets of arbitrary positive measure. This relates to a question posed by J. Palis which arises naturally in the context of homoclinic bifurcations in dimension 2.

**§1. Introduction.** An interesting theorem of Steinhaus [**1**] asserts that if $A$ and $B$ are measurable subsets of some $\mathbb{R}^k$ with positive measure, then their arithmetical difference $A - B$ (hence their sum) contains an open ball. There are many sets of measure zero, however, such that their difference sets also contain an open ball. For example, it is well-known that $K - K = [-1, 1]$, where $K \subset [0, 1]$ is the classical middle-third Cantor set. In fact, the problem of investigating properties of $A - B$ where $A$ and $B$ have measure zero is much more challenging.

In [**2**], J. Palis proposes some problems concerning difference set of Cantor sets on the real line. One of them is:

**Problem**. Let $K_1$ and $K_2$ be two 'affine' (to be defined below) Cantor sets in $\mathbf{R}$. Is it true that $K_1 - K_2$ has measure zero or else contains an interval? Is the same conclusion true for 'dynamically defined' (to be defined below) Cantor sets?

The purpose of this note is to give a general counterexample to the second question above. While completing this paper, I was informed that the same problem is solved independently by Sannami [**3**]. Apparently the first question is still open.

**§2. Preliminaries and Notation.** Let $0 = a_0 < a_1 < \cdots < a_{2m+1} = 1$ be a partition of [0,1] with $m \geq 1$. Set $E_j = [a_{2j}, a_{2j+1}]$, $0 \leq j \leq m$. By definition, a

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*dynamically defined Cantor set* $K$ *with basis* $E_0 \cup \cdots \cup E_m$ is $\bigcap_{n \geq 0} \psi^{-n}(E_0 \cup \cdots \cup E_m)$, where $\psi : E_0 \cup \cdots \cup E_m \longrightarrow [0,1]$ is a $C^{1+\alpha}$ function, $\alpha > 0$, and $\psi|E_j$ is an expanding map onto [0,1]. Therefore a dynamically defined Cantor set is determined by a finite collection of disjoint closed intervals in [0,1] and a $C^{1+\alpha}$ expanding map on this collection onto [0,1].

A dynamically defined Cantor set is called *affine* if the restriction of $\psi$ to each $E_j$ is affine, i.e., $(\psi|E_j)'' = 0$. In other words, an affine Cantor set is obtained by first removing a finite number of open intervals in [0,1], then applying the same surgery on each of the remaining intervals by a *linear* change of scale, and so on.

For our purposes, we shall be primarily concerned with a special class of Cantor sets. Strictly speaking, for every integer $p \geq 1$ and every sequence $\alpha = (\alpha_1, \alpha_2, \cdots)$ of real numbers with $0 < \alpha_j < 1/p$, *the middle p-Cantor set with index* $\alpha$, $K(\alpha, p)$, is defined as follows: First remove from [0,1] its $p$ 'middle' open intervals each of length $\alpha_1$, and denote the remainder by $K_1$. Next remove from each connected component of $K_1$, say $J$, its $p$ 'middle' open intervals each of length $\alpha_2 \cdot m(J)$, and denote the remainder by $K_2$, an so on. Then $K(\alpha, p) := \bigcap_{n \geq 1} K_n$. For example, the classical middle third Cantor set is $K(\alpha, 1)$, where $\alpha = (\frac{1}{3}, \frac{1}{3}, \cdots)$. A nontrivial question about these Cantor sets is to find conditions on the index $\alpha$ under which $K(\alpha, p)$ is dynamically defined. Evidently $K(\alpha, p)$ is affine iff $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \cdots$.

In the next section we first construct a class of $K(\alpha, p)$'s whose self-difference sets have positive measure yet containing no open intervals. Next we determine conditions on the index $\alpha$ such that $K(\alpha, p)$ is dynamically defined. The fact that such $\alpha$'s exist gives a negative answer to the second question mentioned in the first section.

## §3. Main Result.

First of all we prove the following

**Theorem 1.** *Given $p \geq 1$ there exist infinitely many indices $\alpha$ for which the self-difference set of $K = K(\alpha, p)$ is a Cantor set with any given measure $0 < \sigma < 2$.*

**Proof**. That $t \in K - K$ means the line $y = x - t$ in $\mathbb{R}^2$ intersects $K \times K$ (it is evident that $K - K \subset [-1, 1]$). Choose $\alpha$ in such a way that $1/(2p+1) < \alpha_j < 1/p$

for every $j \geq 1$. In each step of constructing $K$, there appear finitely many open intervals in $[-1, 1]$ such that if $t$ belongs to one of these intervals, then the line $y = x - t$ does not intersect $K \times K$. More precisely, in the $n$-th step of construction $K$, there appear $2p(2p+1)^{n-1}$ new open intervals in $[-1, 1]$ such that the line $y = x - t$ does not intersect $K_n \times K_n$ whenever $t$ belongs to one of these intervals. If we remove the union of these intervals from $[-1, 1]$, the remaining Cantor set is exactly $K - K$, for if $t$ belongs to this remainder, then the line $y = x - t$ intersects one of the $(p+1)^{2n}$ squares in $K_n \times K_n$ for every $n \geq 1$.

To compute the measure of $K - K$ we have to know total length of intervals appearing in the $n$-th step. A straightforward computation shows that this total length is

$$\frac{2p}{p+1}\left(\frac{2p+1}{p+1}\right)^{n-1}(1-p\alpha_1)(1-p\alpha_2)\cdots(1-p\alpha_{n-1})((2p+1)\alpha_n - 1),$$

so that

$$m(K-K) = 2 - \frac{2p}{p+1}\sum_{n=1}^{\infty}\left(\frac{2p+1}{p+1}\right)^{n-1}(1-p\alpha_1)\cdots(1-p\alpha_{n-1})((2p+1)\alpha_n - 1). \tag{1}$$

Therefore the problem reduces to the careful determination of the index $\alpha$.

Choose an arbitrary sequence $\{\gamma_n\}$ such that $0 < \gamma_n < \left(\frac{\sigma}{2}\right)^{n-1}(2-\sigma)$ for $n \geq 2$ and $\sum_{n=1}^{\infty} \gamma_n = 2 - \sigma$. Define

$$\alpha_1 := \frac{p+1}{2p(2p+1)}\gamma_1 + \frac{1}{2p+1}.$$

Then

$$\frac{1}{2p+1} < \alpha_1 < \frac{(p+1)(2-\sigma)}{2p(2p+1)} + \frac{1}{2p+1} = \frac{1}{p} - \frac{\sigma(p+1)}{2p(2p+1)},$$

so that $(1-p\alpha_1)^{-1} < \dfrac{2(2p+1)}{\sigma(p+1)}$. Having defined $\alpha_1, \cdots, \alpha_{n-1}$, put

$$\alpha_n := \frac{1}{2p}\left(\frac{p+1}{2p+1}\right)^n (1-p\alpha_1)^{-1}\cdots(1-p\alpha_{n-1})^{-1}\gamma_n + \frac{1}{2p+1}. \tag{2}$$

Then we have

$$\frac{1}{2p+1} < \alpha_n \quad < \quad \frac{1}{2p}\left(\frac{p+1}{2p+1}\right)^n \left(\frac{2(2p+1)}{\sigma(p+1)}\right)^{n-1} \left(\frac{\sigma}{2}\right)^{n-1}(2-\sigma) + \frac{1}{2p+1}$$
$$= \frac{(p+1)(2-\sigma)}{2p(2p+1)} + \frac{1}{2p+1},$$

so that

$$\frac{1}{2p+1} < \alpha_n < \frac{1}{p} - \frac{\sigma(p+1)}{2p(2p+1)}. \tag{3}$$

Thus the inductive definition (2) proceeds in such a way that (3) always holds. From (2) it is clear that

$$\frac{2p}{p+1}\left(\frac{2p+1}{p+1}\right)^{n-1}(1-p\alpha_1)\cdots(1-p\alpha_{n-1})\left((2p+1)\alpha_n - 1\right) = \gamma_n,$$

so that $m(K-K) = \sigma$ by (1) and the choice of $\gamma_n$. $\square$

**Remarks.**

(i) For an affine Cantor set of type $K(\alpha, p)$ one must have $\alpha_j = c$ for $j \geq 1$. From (1) it can be easily seen that in such a case (whenever $1/(2p+1) < c < 1/p$) we have $m(K-K) = 0$, so this construction cannot yield an affine Cantor set with self-difference of positive measure. Similarly, $m(K-K) = 0$ even if we impose the condition $\alpha_j = c$ for $j \geq$ some $n$.

(ii) The Cantor set constructed in the above proof has Hausdorff dimension$< 1$, which is a necessary condition for $K$ to be dynamically defined. In fact if $\epsilon > 0$ and $N_\epsilon(K)$ is the minimal number of intervals of length $\epsilon$ needed to cover $K$, then one has [**2**]

$$HD(K) \leq \liminf_{\epsilon \to 0} \frac{\log N_\epsilon(K)}{-\log \epsilon}.$$

But by the construction it is clear that $N_{\epsilon_n}(K) = (p+1)^n$, where $\epsilon_n = \prod_{j=1}^n \left(\frac{1-p\alpha_j}{p+1}\right)$. Therefore

$$\begin{aligned} HD(K) &\le \liminf_{n\to\infty} \frac{\log N_{\epsilon_n}(K)}{-\log \epsilon_n} \\ &= \liminf_{n\to\infty} \frac{n\log(p+1)}{n\log(p+1) - \sum_{j=1}^n \log(1-p\alpha_j)} \\ &\le \liminf_{n\to\infty} \frac{n\log(p+1)}{n\log(p+1) - n\log\left(\frac{p+1}{2p+1}\right)} \\ &= \frac{\log(p+1)}{\log(2p+1)}. \end{aligned}$$

However, for $K$ to be dynamically defined we need $\alpha_n$ to converge very fast to $1/(2p+1)$. This is possible by the following

**Lemma 1.** *The index $\alpha$ of $K(\alpha, p)$ in Theorem 1 can be chosen such that $\alpha_n$ decreases geometrically to $1/(2p+1)$ as fast as we desire.*

**Proof**. Choose the sequence $\{\gamma_n\}$ such that $0 < \gamma_n < \left(\frac{\sigma}{2}\right)^{\nu(n-1)}(2-\sigma)$ for $n \ge 2$, where $\nu$ is an arbitrary real $> 1$, $\gamma_n < \frac{\sigma}{2}\gamma_{n-1}$, and $\sum_{n=1}^\infty \gamma_n = 2-\sigma$. Then the index $\alpha$ can be constructed by exactly the same method so that (2) and (3) still hold, but now (2) shows that

$$\begin{aligned} \alpha_n - \frac{1}{2p+1} &= \frac{1}{2p}\left(\frac{p+1}{2p+1}\right)^n (1-p\alpha_1)^{-1}\cdots(1-p\alpha_{n-1})^{-1}\gamma_n \\ &\le \frac{1}{2p}\left(\frac{p+1}{2p+1}\right)^n \cdot \left(\frac{2(2p+1)}{\sigma(p+1)}\right)^{n-1} \cdot \left(\frac{\sigma}{2}\right)^{\nu(n-1)}(2-\sigma) \qquad \text{(by(3))} \qquad (4) \\ &= \frac{1}{2p}\cdot\frac{p+1}{2p+1}\left(\frac{\sigma}{2}\right)^{(\nu-1)(n-1)}(2-\sigma) \end{aligned}$$

and the rate of convergence can be controlled by $\nu$ since $0 < \sigma < 2$. Finally (2) and (3) show that $\alpha_n < \alpha_{n-1}$ since $\gamma_n < \frac{\sigma}{2}\gamma_{n-1}$. $\square$

Now the main result can be stated in

**Theorem 2.** *Among the Cantor sets constructed in Theorem 1, there are infinitely many dynamically defined ones whose self-difference sets have given measure* $0 < \sigma < 2$.

The key idea in proving Theorem 2 is that by Lemma 1 one can arrange things so that $\alpha_n$ converges very fast to $1/(2p+1)$. If the rate of convergence is large enough, i.e., if $K(\alpha, p)$ is closely similar to affine $K(\alpha', p)$ with $\alpha' = \left(\frac{1}{2p+1}, \frac{1}{2p+1}, \dots\right)$, then this $K(\alpha, p)$ is dynamically defined and its $\psi$ is closely similar to the affine map $t \mapsto (2p+1)t$ on $K(\alpha, p)$.

Before proving the above theorem we need two lemmas.

**Lemma 2.** *Suppose that* $\epsilon > 0$, $\beta > 2p+1$. *Then there exists a* $C^\infty$ *function* $f = f_{\beta,\epsilon} : [0, \epsilon] \to [0, \beta\epsilon]$ *such that*

(i) $f(0) = 0$ *and* $f(\epsilon) = \beta\epsilon$,

(ii) $f'(t) \geq 2p+1$,

(iii) $f'(0) = f'(\epsilon) = 2p+1$ *and* $f^{(n)}(0) = f^{(n)}(\epsilon) = 0$ *for* $n \geq 2$,

(iv) $(2p+1)t \leq f(t) \leq (2p+1)t + (\beta - (2p+1))\,\epsilon$.

**Proof.** For $t \in [0, \epsilon]$ define

$$h(t) = h_{\beta,\epsilon}(t) := \exp\left\{-\left(\frac{1}{t} + \frac{1}{\epsilon - t}\right)\right\},$$

$$g(t) = g_{\beta,\epsilon}(t) := \int_0^t h(\tau)d\tau \Big/ \int_0^\epsilon h(\tau)d\tau, \tag{5}$$

and

$$f(t) = f_{\beta,\epsilon}(t) := (2p+1)t + (\beta - (2p+1))\,\epsilon g(t). \tag{6}$$

Note that $g'(t) \geq 0$ , $g^{(n)}(0) = g^{(n)}(\epsilon) = 0$ for $n \geq 1$, and $0 \leq g(t) \leq 1$. Now it is easily verified that $f$ has the desired properties. $\square$

**Lemma 3.** *With the above notation, if* $0 < \epsilon < 1$, *we have*

(i) $\displaystyle\int_0^\epsilon h(t)dt \geq (\epsilon^2/30)e^{-4/\epsilon}$,

(ii) $\sup_{t\in[0,\epsilon]} |h'(t)| \leq (64/\epsilon^2)e^{-4/\epsilon}$.

**Proof.** (i) Let

$$\tilde{h}(t) := \begin{cases} (30/\epsilon^2)e^{-4/\epsilon}\Big(t - (\frac{\epsilon}{2} - \frac{\epsilon^2}{30})\Big) & t \in [\frac{\epsilon}{2} - \frac{\epsilon^2}{30}, \frac{\epsilon}{2}] \\ 0 & t \in [0, \frac{\epsilon}{2} - \frac{\epsilon^2}{30}]. \end{cases}$$

Note that if $t \in [\frac{\epsilon}{2} - \frac{\epsilon^2}{30}, \frac{\epsilon}{2}]$, then

$$\frac{1}{t^2} - \frac{1}{(\epsilon - t)^2} = \frac{\epsilon(\epsilon - 2t)}{\big(t(\epsilon - t)\big)^2} < \frac{\epsilon^2}{(\frac{\epsilon}{4}.\frac{3\epsilon}{4})^2} = \frac{(16)^2}{9}.\frac{1}{\epsilon^2} < \frac{30}{\epsilon^2}.$$

Therefore $\frac{d}{dt}\Big(h(t) - \tilde{h}(t)\Big) \leq 0$ on $[\frac{\epsilon}{2} - \frac{\epsilon^2}{30}, \frac{\epsilon}{2}]$, so that $h(t) \geq \tilde{h}(t)$ on $[0, \frac{\epsilon}{2}]$. Hence one has

$$\begin{aligned} \int_0^\epsilon h(t)dt &= 2\int_0^{\epsilon/2} h(t)dt \\ &\geq 2\int_{\epsilon/2-\epsilon^2/30}^{\epsilon/2} \tilde{h}(t)dt \\ &= \epsilon^2/30\ e^{\,-4/\epsilon}. \end{aligned}$$

(ii) First we show that for $t \in [0, \frac{\epsilon}{8}]$, $h''(t) \geq 0$. This is equivalent to $\left(\frac{1}{t^2} - \frac{1}{(\epsilon - t)^2}\right)^2 - \frac{2}{t^3} - \frac{2}{(\epsilon - t)^3} \geq 0$, which in turn is equivalent to $\epsilon(\epsilon - 2t)^2 \geq 2t(\epsilon - t)(\epsilon^2 + 3t^2 - 3\epsilon t)$. But for $t \in [0, \frac{\epsilon}{8}]$ one has

$$\epsilon(\epsilon - 2t)^2 \geq \frac{9}{16}\epsilon^3 \geq \frac{67}{256}\epsilon^4 \geq 2t(\epsilon - t)(\epsilon^2 + 3t^2 - 3\epsilon t).$$

This being so, we have

$$\sup_{t\in[0,\epsilon/8]} h'(t) = h'(\frac{\epsilon}{8}) = \left(\frac{64}{\epsilon^2} - \frac{64}{49\epsilon^2}\right)e^{-\left(\frac{8}{\epsilon} + \frac{8}{7\epsilon}\right)}.$$

On the other hand, on $[\frac{\epsilon}{8}, \frac{\epsilon}{2}]$ we have

$$0 \leq h'(t) \leq \left(\frac{64}{\epsilon^2} - \frac{64}{49\epsilon^2}\right)e^{-4/\epsilon}.$$

Since $h(t) = h(\epsilon - t)$, the result follows. $\square$

Now we are ready to prove Theorem 2. Recall that we can write $K = K(\alpha, p)$ as $K = \bigcap_{n \geq 0} K_n$, where $K_0 = [0, 1]$ and $K_n$ is defined as in section 2.

**Proof of Theorem 2.** Fix $p \geq 1$ and consider a $K(\alpha, p)$ as in Theorem 1. By Lemma 1 we can select the index $\alpha$ so that $\alpha_n$ decreases geometrically to $1/(2p+1)$ at a rate $\nu > 1$ to be carefully chosen later.

Decompose $K_n$ from the left to the right into its connected components $K_n^1, K_n^2, \cdots, K_n^{(p+1)^n}$, where each $K_n^j := [\lambda_{n,j}, \mu_{n,j}]$ is a closed interval of length $\epsilon_n := \prod_{j=1}^n \left(\frac{1 - p\alpha_j}{p+1}\right)$. Similarly, decompose $K_{n-1} \backslash K_n$ from the left to the right into its connected components $H_n^1, H_n^2, \ldots, H_n^{p(p+1)^{n-1}}$, where each $H_n^j$ is an open interval of length $\epsilon'_n := \alpha_n \epsilon_{n-1}$. Set $\beta_n := \left(\frac{p+1}{1 - p\alpha_n}\right)$ and $\zeta_n := \left(\frac{\alpha_{n-1}}{\alpha_n}\right) \beta_{n-1}$. From (3) and Lemma 1 it is clear that

$$2p + 1 < \beta_n < \frac{2}{\sigma}(2p+1) \ , \ \zeta_n > \beta_n. \tag{7}$$

We want to show that $K$ is a dynamically defined Cantor set with basis $K_1^1, K_1^2, \cdots, K_1^{p+1}$. Evidently it suffices to construct $\psi$ on $K_1^1$, for then we can repeat this $\psi$ on every $K_1^j$. Define a sequence $\{\psi_n\}_{n \geq 1}$ of $C^\infty$ functions on $K_1^1 = [0, \epsilon_1]$ as follows: Set $\psi_1 = f_{\beta_1, \epsilon_1}$ on $K_1^1$,

$$\psi_2(t) = \begin{cases} f_{\beta_2, \epsilon_2}(t - \lambda_{2,j}) + \lambda_{1,j} & t \in K_2^j, \quad 1 \leq j \leq p+1 \\ f_{\zeta_2, \epsilon'_2}(t - \mu_{2,j}) + \mu_{1,j} & t \in H_2^j, \quad 1 \leq j \leq p, \end{cases}$$

and define $\psi_n$ for $n \geq 3$ as

$$\psi_n(t) = \begin{cases} \psi_{n-1}(t) & t \in K_1^1 \backslash K_{n-1} \\ f_{\beta_n, \epsilon_n}(t - \lambda_{n,j}) + \lambda_{n-1,j} & t \in K_n^j, \qquad 1 \leq j \leq (p+1)^{n-1} \\ f_{\zeta_n, \epsilon'_n}(t - \mu_{n,j}) + \mu_{n-1,j} & t \in H_n^j, \qquad 1 \leq j \leq p(p+1)^{n-2}. \end{cases} \tag{8}$$

Clearly each $\psi_n$ maps $K_1^1$ onto [0,1] in a $C^\infty$ way, $\psi_n$ is strictly increasing, $\psi'_n(t) \geq 2p + 1$, and

$$\psi_m(K_n^j) = K_{n-1}^j \ \text{ for } 1 \leq j \leq (p+1)^{n-1} \text{ and } \ m \geq n, \tag{9}$$

$$\psi_m(H_n^j) = H_{n-1}^j \ \text{ for } 1 \leq j \leq p(p+1)^{n-2} \text{ and } \ m \geq n. \tag{10}$$

Fix some $n \geq 2$. We are going to estimate the difference $|\psi''_{n+1} - \psi''_n|$ on $K_1^1$. Since $\psi''_{n+1} = \psi''_n$ on $K_1^1 \backslash K_n$ by (8), it suffices to estimate this difference on $K_n \cap K_1^1$. Evidently

$$\sup_{t \in K_n \cap K_1^1} |\psi''_{n+1}(t) - \psi''_n(t)| = \sup_{t \in K_n^1} |\psi''_{n+1}(t) - \psi''_n(t)|.$$

But for $t \in K_n^1 = [0, \epsilon_n]$ we have

$$\begin{aligned}
|\psi''_n(t)| &= |f''_{\beta_n,\epsilon_n}(t)| && (by (8)) \\
&= (\beta_n - (2p+1))\epsilon_n |g''_{\beta_n,\epsilon_n}(t)| && (by(6)) \\
&= \frac{(\beta_n - (2p+1))\,\epsilon_n}{\int_0^{\epsilon_n} h_{\beta_n,\epsilon_n}(t)dt} |h'_{\beta_n,\epsilon_n}(t)| && (by(5)) \\
&\leq \frac{(\beta_n - (2p+1))\,\epsilon_n}{(\epsilon_n^2/30)e^{-4/\epsilon_n}} \cdot \frac{64}{\epsilon_n^2} e^{-4/\epsilon_n} && (by\ lemma(3)) \\
&= \frac{(\text{const.})\,(\beta_n - (2p+1))}{\epsilon_n^3}.
\end{aligned} \tag{11}$$

On the other hand, $\epsilon_n = \prod_{j=1}^n \beta_j^{-1} \geq \left(\frac{\sigma}{2}\right)^n (2p+1)^{-n}$ by (7), and

$$\begin{aligned}
\beta_n - (2p+1) &= \beta_n \left(1 - \frac{(2p+1)(1-p\alpha_n)}{p+1}\right) \\
&\leq \frac{2}{\sigma} \frac{p(2p+1)}{p+1} \left((2p+1)\alpha_n - 1\right) \\
&\leq \frac{(2p+1)}{\sigma} \left(\frac{\sigma}{2}\right)^{\nu(n-1)} (2-\sigma). \qquad (\text{by}(4))
\end{aligned}$$

Hence we have

$$\begin{aligned}
\sup_{t \in K_n^1} |\psi''_n t)| &\leq (\text{const.}) \left(\frac{\sigma}{2}\right)^{\nu(n-1)-3n} (2p+1)^{3n} \\
&\leq (\text{const.}) \left(\frac{\sigma}{2}\right)^{(\nu-3)n} (2p+1)^{3n}.
\end{aligned} \tag{12}$$

Now we estimate $|\psi''_{n+1}|$ on $K^1_n$. First suppose that $t \in K^1_{n+1}$. Then

$$|\psi''_{n+1}(t)| = |f''_{\beta_{n+1},\epsilon_{n+1}}(t)| \leq (\text{const.}) \left(\frac{\sigma}{2}\right)^{(\nu-3)n} (2p+1)^{3n} \tag{13}$$

by (12). Next suppose that $t \in H^1_{n+1}$. Then since $\alpha_n > \alpha_{n+1} > 1/(2p+1)$,

$$\begin{aligned}
|\psi''_{n+1}(t)| &= |f''_{\zeta_{n+1},\epsilon'_{n+1}}(t)| \\
&\leq \frac{(\text{const.})\,(\zeta_{n+1} - (2p+1))}{(\epsilon'_{n+1})^3} && (by(11)) \\
&= \frac{(\text{const.})}{\alpha^3_{n+1}\epsilon^3_n}.\frac{\alpha_n}{\alpha_{n+1}}\left(\beta_n - \frac{\alpha_{n+1}}{\alpha_n}(2p+1)\right) \\
&\leq \frac{(\text{const.})}{\epsilon^3_n}(\beta_n - (2p+1)) + \frac{(\text{const.})}{\epsilon^3_n}(2p+1)\left(1 - \frac{\alpha_{n+1}}{\alpha_n}\right) && (14) \\
&\leq (\text{const.}) \left(\frac{\sigma}{2}\right)^{(\nu-3)n} (2p+1)^{3n}.
\end{aligned}$$

From (13) and (14) we have

$$\begin{aligned}
\sup_{t\in K^1_n} |\psi''_{n+1}(t)| &= \max\left\{\sup_{t\in K^1_{n+1}} |\psi''_{n+1}(t)|, \sup_{t\in H^1_{n+1}} |\psi''_{n+1}(t)|\right\} \\
&\leq (\text{const.}) \left(\frac{\sigma}{2}\right)^{(\nu-3)n} (2p+1)^{3n}. && (15)
\end{aligned}$$

Now take $\nu > 1$ so large that $\tau := \left(\frac{\sigma}{2}\right)^{\nu-3}(2p+1)^3 < 1$. Two estimates (12) and (15) will then show that

$$\sup_{t\in K^1_1} |\psi''_{n+1}(t) - \psi''_n(t)| \leq (\text{const.})\ \tau^n.$$

However this means that $\{\psi''_n\}$ is uniformly convergent. Since $\{\psi'_n\}$ converges in at least one point (say $t$=0), we conclude that $\{\psi'_n\}$ is uniformly convergent. Again since $\{\psi_n\}$ converges in at least one point, $\{\psi_n\}$ will converge uniformly on $K^1_1$ to a mapping $\psi$ which is clearly (at least) $C^2$. Moreover, $\psi$ maps $K^1_1$ onto [0,1] and $\psi'(t) \geq 2p+1$. Repeating this $\psi$ on every $K^j_1$ we obtain the required mapping (also denoted by $\psi$). Finally (9) and (10) show that $\psi(K^j_n) = K^j_{n-1}$ for $1 \leq j \leq (p+1)^n$

and $\psi(H_n^j) = H_{n-1}^j$ for $1 \leq j \leq p(p+1)^{n-1}, n \geq 1$. From this it can be checked that $\psi^{-n}(K_1) = K_{n+1}$, and we are done. $\square$

**§4. Final Remark.** While the above proof gives sufficient conditions on $K(\alpha, p)$ to be dynamically defined, it is interesting to answer the following

**Problem.** Given $p \geq 1$, find necessary and sufficient conditions on the index $\alpha$ which guarantee $K(\alpha, p)$ is dynamically defined.

## References

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