Contemporary Mathematics

Dynamics of Singular Holomorphic Foliations on the Complex Projective Plane

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Preface

This manuscript is a revised version of my Master's thesis which was originally written in 1992 and was presented to the Mathematics Department of University of Tehran. My initial goal was to give, in a language accessible to non-experts, highlights of the 1978 influential paper of Il'yashenko on singular holomorphic foliations on $\mathbb{CP}^2$ [**I3**], providing short, self-contained proofs. Parts of the exposition in chapters 1 and 3 were greatly influenced by the beautiful work of Gómez-Mont and Ortiz-Bobadilla [**GO**] in Spanish, which contains more material, different from what we discuss here. It must be noted that much progress has been made in this area since 1992, especially in local theory (see for instance the collection [**I6**] and the references cited there). However, Hilbert's 16th Problem and the Minimal Set Problem are still unsolved.

There is a well-known connection between holomorphic foliations in dimension 2 and dynamics of iterations of holomorphic maps in dimension 1, but many believe that this connection has not been fully exploited. It seems that some experts in each area keep an eye on progress in the other, but so far there have been rather few examples of a fruitful interaction. The conference on Laminations and Foliations held in May 1998 at Stony Brook was a successful attempt to bring both groups together. As a result, many people in dynamics expressed their interest in learning

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about holomorphic foliations. I hope the present manuscript will give them a flavor of the subject and will help initiate a stronger link between the young researchers in both areas.

My special thanks go to S. Shahshahani who encouraged me to study this subject years ago and supervised my Master's thesis back in Iran. I gratefully acknowledge the financial support of IPM (Tehran) during preparation of this manuscript. A series of lectures given by C. Camacho and Yu. Il'yashenko in Trieste, Italy, in 1991 and 1992 were a source of inspiration to me. I am indebted to X. Gómez-Mont and J. Milnor who read parts of this revised version and made very useful comments. Finally, I would like to thank the Dynamical Systems group at Stony Brook, especially A. Epstein, M. Lyubich and J. Milnor. Without the interest they showed, I would have never found enough motivation to revise what I had written 6 years ago.

Introduction

Consider the differential equation

$$(*)\begin{cases} \dfrac{dx}{dt} = P(x,y) \\[2mm] \dfrac{dy}{dt} = Q(x,y) \end{cases}$$

in the real plane $(x, y) \in \mathbb{R}^2$, where P and Q are relatively prime polynomials with $\max\{\deg P, \deg Q\} = n$. What can be said, Hilbert asked, about the number and the location of limit cycles of $(*)$? In particular, is it true that there are only finitely many limit cycles for $(*)$? If so, does there exist an upper bound $H(n)$, depending only on n, for the number of limit cycles of an equation of the form $(*)$? Surprisingly, the finiteness problem has been settled only in recent years, and the existence and a possible value of $H(n)$ is still unknown, even when $n = 2$!

In an attempt to answer the Hilbert's question, H. Dulac "proved" the finiteness theorem in 1926 [**D**]. Many years later, however, his "proof" turned out to be wrong. In fact, in 1982 Yu. Il'yashenko found a fundamental mistake in Dulac's argument [**I2**], and gave a correct proof for the finiteness theorem later in 1987 [**I4**].

The second major attempt along this line was started in 1956 by a seminal paper of I. Petrovskiĭ and E. Landis [**PL1**]. They had a completely different and perhaps more radical approach. They considered $(*)$ as a differential equation in the complex plane $(x, y) \in \mathbb{C}^2$, with t now being a *complex time* parameter. The integral curves of the vector field are now either singular points which correspond to the common zeros of P and Q, or complex curves tangent to the vector field which are holomorphically immersed in $\mathbb{C}^2$. This gives rise to a holomorphic foliation by complex curves with a finite number of singular points. One can easily see that this foliation extends to the complex projective plane $\mathbb{CP}^2$, which is obtained by adding a line at infinity to the plane $\mathbb{C}^2$. The trajectories of $(*)$ in the real plane are then the intersection of these complex curves with the plane $\text{Im } x = \text{Im } y = 0$.

What makes this approach particularly useful is the possibility of applying methods of several complex variables and algebraic geometry over an algebraically closed field, not available in the real case. Intuitively, the complexified equation provides enough space to go around and observe how the integral curves behave, whereas the real-plane topology of the trajectories is only the tip of a huge iceberg.

Viewing $(*)$ as a complex differential equation, Petrovskiĭ and Landis "proved" that $H(2)$ exists and in fact one can take $H(2) = 3$ [**PL1**]. Later on, they "proved" the estimates

$$H(n) \leq \begin{cases} \dfrac{6n^3 - 7n^2 + n + 4}{2} & \text{if } n \text{ is even,} \\ \dfrac{6n^3 - 7n^2 - 11n + 16}{2} & \text{if } n \text{ is odd,} \end{cases}$$

thus answering Hilbert's question [**PL2**]. The result was regarded as a great achievement: Not only did they solve a difficult problem, but they introduced a truly novel method in the geometric theory of ordinary differential equations. However, it did not take long until a serious gap was discovered in their proof. Although the estimate on $H(n)$ could not be salvaged anymore, the powerful method of this work paved the way for further studies in this direction.

In 1978, Il'yashenko made a fundamental contribution to the problem. Following the general idea of Petrovskiĭ and Landis, he studied equations $(*)$ with complex polynomials P, Q from a topological standpoint without particular attention to Hilbert's question. In his famous paper [**I3**], he showed several peculiar properties of the integral curves of such equations.

From the point of view of foliation theory, it may seem that foliations induced by equations like $(*)$ form a rather tiny class among all holomorphic foliations on $\mathbb{CP}^2$. However, as long as we impose a reasonable condition on the set of singularities, it turns out that *every* singular holomorphic foliation on $\mathbb{CP}^2$ is induced by a polynomial differential equation of the form $(*)$ in the affine chart $(x, y) \in \mathbb{C}^2$. The condition on singularities is precisely what is needed in several complex variables: the singular set of the foliation must be an analytic subvariety of codimension > 1, which is just a finite set in the case of the projective plane.

Consider a closed orbit γ of a smooth vector field in the real plane. To describe the behavior of trajectories near γ, one has the simple and useful concept of the *Poincaré first return map*: Choose a small transversal Σ at some point $p \in \gamma$, choose a point $q \in \Sigma$ near p, and look at the first point of intersection with Σ of the trajectory passing through q. In this way, one obtains the germ of a smooth diffeomorphism of Σ fixing p. The iterative dynamics of this self-map of Σ reflects the global behavior of the trajectories near γ.

For a singular holomorphic foliation on $\mathbb{CP}^2$ induced by an equation of the form $(*)$, a similar notion, called the *monodromy mapping*, had already been used by Petrovskiĭ and Landis, and extensively utilized by Il'yashenko. A closed orbit should now be replaced by a non-trivial loop γ on the leaf passing through a given point p, small transversal Σ is a 2-disk, and the result of traveling over γ on the leaf passing through a point on Σ near p gives the germ of a biholomorphism of Σ fixing p, called the *monodromy mapping associated with* γ. Note that all points in the orbit of a given point on Σ under this biholomorphism lie in the same leaf. In this way, to each non-trivial loop in the fundamental group of the leaf we associate a self-map of Σ reflecting the behavior of nearby leaves as one goes around the loop. It is easily checked that composition of loops corresponds to superposition of monodromy mappings, so the fundamental group of the leaf maps homomorphically to a subgroup of the group of germs of biholomorphisms of Σ fixing p; the latter subgroup will be called the *monodromy group* of the leaf. Thus a global problem can be reduced to a large extent to the study of germs of biholomorphisms of $\mathbb{C}$ fixing (say) the origin. The group of all these germs is denoted by $\mathrm{Bih}_0(\mathbb{C})$.

Now here is the crucial observation: For "almost every" equation of the form $(*)$, the line at infinity of $\mathbb{CP}^2$ with finitely many singular points deleted is a leaf of the extended foliation. On the other hand, no leaf can be bounded in $\mathbb{C}^2$ (this is in fact more than just the Maximum Principle), so every leaf has a point of accumulation on the line at infinity. Therefore, the monodromy group of the *leaf at infinity*, which is finitely-generated since the leaf is homeomorphic to a finitely-punctured Riemann sphere, gives us much information about the global behavior of all leaves.

What Il'yashenko observed was the fact that almost all the dynamical properties of leaves have a discrete interpretation in terms of finitely-generated subgroups of $\mathrm{Bih}_0(\mathbb{C})$, the role of which is played by the monodromy group of the leaf at infinity. Therefore, he proceeded to study these subgroups and deduced theorems about the behavior of leaves of singular holomorphic foliations. The density and ergodicity theorems for these foliations are direct consequences of the corresponding results for subgroups of $\mathrm{Bih}_0(\mathbb{C})$. The density theorem asserts that for "almost every" equation $(*)$, every leaf other than the leaf at infinity is dense in $\mathbb{CP}^2$. The ergodicity theorem says that "almost every" foliation induced by an equation of the form $(*)$ is *ergodic*, which means that every measurable saturated subset of $\mathbb{CP}^2$ has zero or full measure.

Yet another advantage of the reduction to the discrete case is the possibility of studying consequences of equivalence between two such foliations. Two singular holomorphic foliations on $\mathbb{CP}^2$ are said to be *equivalent* if there exists a homeomorphism of $\mathbb{CP}^2$ which sends each leaf of the first foliation to a leaf of the second one. Such an equivalence implies the equivalence between the monodromy groups of the corresponding leaves at infinity. The latter equivalence has the following meaning: $G, G' \subset \mathrm{Bih}_0(\mathbb{C})$ are *equivalent* if there exists a homeomorphism h of some neighborhood of $0 \in \mathbb{C}$, with $h(0) = 0$, such that $h \circ f \circ h^{-1} \in G'$ if and only if $f \in G$. The correspondence $f \mapsto h \circ f \circ h^{-1}$ is clearly a group isomorphism. Under typical conditions, one can show that the equivalence between two subgroups of $\mathrm{Bih}_0(\mathbb{C})$ must in fact be holomorphic. This gives some invariants of the equivalence classes, and reveals a rigidity phenomenon. Applying these results to the monodromy groups of the leaves at infinity, one can show the existence of *moduli of stability* and the phenomenon of *absolute rigidity* for these foliations.

All the above results are proved under typical conditions on the differential equations. After all, the existence of the leaf at infinity (an algebraic leaf) plays a substantial role in all these arguments. Naturally, one would like to study foliations which do not satisfy these conditions, in particular, those which do not admit any algebraic leaf. From this point of view, the dynamics of these foliations is far from being understood. The major contributions along this line have been made by the Brazilian and French schools. In their interesting paper [**CLS1**], C. Camacho, A. Lins Neto and P. Sad study non-trivial minimal sets of these foliations and show several properties of the leaves within a non-trivial minimal set. A *minimal set* of a foliation on $\mathbb{CP}^2$ is a compact, saturated, non-empty subset of $\mathbb{CP}^2$ which is minimal with respect to these three properties. A *non-trivial minimal set* is one which is not a singular point. It follows that the existence of a non-trivial minimal set is equivalent to the existence of a leaf which does not accumulate on any singular point. The fact that non-trivial minimal sets do not exist when the foliation admits an algebraic leaf makes the problem much more challenging. The major open problem in this context is the most primitive one: "Does there exist a singular holomorphic foliation on $\mathbb{CP}^2$ having a non-trivial minimal set?"

An intimately connected question is about limit sets of the leaves of these foliations. The classical theorem of Poincaré−Bendixson classifies all possible limit sets for foliations on the 2-sphere: Given a smooth vector field on the 2-sphere with a finite number of singular points, the ω- (or α-) limit set of any point is either a singular point or a closed orbit or a chain of singular points and trajectories starting from one of these singular points ending at another. This enables us to understand the asymptotic behavior of *all* trajectories. Naturally, one is interested in proving a complex version of the Poincaré−Bendixson Theorem for foliations on $\mathbb{CP}^2$. Now the concept of the limit set is defined as follows: Take a leaf $\mathcal{L}$ and let $K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$ be a sequence of compact subsets of $\mathcal{L}$, with $\bigcup_{n\geq 1} K_n = \mathcal{L}$. Then the *limit set* of $\mathcal{L}$ is by definition the intersection $\bigcap_{n\geq 1} \overline{\mathcal{L} \smallsetminus K_n}$. Incidentally, the limit set of every non-singular leaf is non-vacuous, since it can be shown that no non-singular leaf can be compact. Despite some results in this direction (see e.g. [**CLS2**]), the problem of classifying possible limit sets is almost untouched.

Finally, it should be mentioned how Hilbert's question for a real equation $(*)$ is interpreted in the complex language. It is easy to prove that a limit cycle of a real equation $(*)$ is a homotopically non-trivial loop on the corresponding leaf of the complexified equation. So any piece of information about the fundamental groups of leaves could be a step toward understanding the limit cycles. A non-trivial loop on a leaf of a complex equation $(*)$ is said to be a *(complex) limit cycle* if the germ of its associated monodromy is not the identity map. Il'yashenko has shown that "almost every" equation $(*)$ has a countably infinite number of homologically independent (complex) limit cycles; nevertheless this result does not have a direct bearing on Hilbert's question. A complex version of the finiteness problem may be the following: Can the fundamental group of a (typical) leaf of such foliations be infinitely-generated? If not, does there exist an upper bound, depending only on the degree of P and Q in $(*)$, for the number of generators of the fundamental groups? Such questions, as far as I know, have not yet been answered.

1. Singular Holomorphic Foliations by Curves

This chapter introduces the concept of a singular holomorphic foliation by (complex) curves on a complex manifold, which will be quite essential in subsequent chapters. We will assume that the dimension of the singular set is small enough to allow easy application of several complex variables techniques. Moreover, when the underlying manifold is $\mathbb{CP}^2$ (2-dimensional complex projective space), the very special geometry of the space allows us to apply some standard algebraic geometry to show that all such foliations are induced by polynomial 1-forms ω on $\mathbb{CP}^2$, the leaves being the solutions of $\omega = 0$. The main tools here are extension theorems of several complex variables and the rigidity properties of holomorphic line bundles on projective spaces.

1.1. Holomorphic Foliations on Complex Manifolds. Let us start with the most basic definition in this subject.

Definition 1.1. Let M be a complex manifold of dimension n and $0 < m < n$. A *(non-singular) holomorphic foliation $\mathcal{F}$ of codimension m* on M is an analytic atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ for M which is maximal with respect to the following properties:

(i) For each $i \in I$, φ_i is a biholomorphism $U_i \to A_i \times B_i$, where A_i and B_i are open polydisks in $\mathbb{C}^{n-m}$ and $\mathbb{C}^m$, respectively.

(ii) If (U_i, φ_i) and (U_j, φ_j) are in $\mathcal{A}$ with $U_i \cap U_j \neq \emptyset$, then $\varphi_{ij} := \varphi_i \circ \varphi_j^{-1}$: $\varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ has the form

$$\varphi_{ij}(z, w) = (\psi_{ij}(z, w), \eta_{ij}(w)), \tag{1.1}$$

where $(z, w) \in \mathbb{C}^{n-m} \times \mathbb{C}^m$, and ψ_{ij} and η_{ij} are holomorphic mappings into $\mathbb{C}^{n-m}$ and $\mathbb{C}^m$, respectively.

Condition (ii) may also be expressed in the following way: Using the coordinates $(z_1, \ldots, z_n)$ for $\mathbb{C}^n$, the mapping $\varphi_{ij} : (z_1, \ldots, z_n) \mapsto (\varphi^1_{ij}, \ldots, \varphi^n_{ij})$ is required to satisfy $\partial\varphi^k_{ij}/\partial z_l = 0$ for $n - m + 1 \leq k \leq n$ and $1 \leq l \leq n - m$.

Each $(U_i, \varphi_i) \in \mathcal{A}$ is called a *foliation chart* (or a *flow box*) for $\mathcal{F}$. Given any foliation chart (U_i, φ_i), the sets $\varphi_i^{-1}(A_i \times \{w\}), w \in B_i$, are called *plaques* of $\mathcal{F}$ in U_i. Evidently, the plaques form a partition of the U_i into connected pieces of complex submanifolds of dimension $n - m$. Each $p \in M$ lies in at least one plaque. Two points p and q are called *equivalent* if there exists a sequence $P_1, \ldots, P_k$ of plaques such that $p \in P_1, q \in P_k$, and $P_i \cap P_{i+1} \neq \emptyset, 1 \leq i \leq k - 1$. The *leaf* of $\mathcal{F}$ through p, denoted by $\mathcal{L}_p$, is the equivalence class of p under this relation. Each leaf has a natural structure of a connected $(n - m)$-dimensional complex manifold which is holomorphically immersed in M. Two leaves are disjoint or else identical.

Remark 1.2. A holomorphic foliation of codimension $(n - 1)$ on a complex manifold M of dimension n is also called a *holomorphic foliation by curves*. Its leaves are immersed Riemann surfaces. As we will discuss later, the field of complex lines tangent to the leaves of such a foliation determines a holomorphic line bundle on M.

Remark 1.3. It can be easily checked that the above definition is equivalent to the following, which is more natural from the geometric viewpoint: A holomorphic foliation $\mathcal{F}$ of codimension m on an n-dimensional complex manifold M is a partition of M into disjoint connected subsets $\{\mathcal{L}_\alpha\}$ (called the *leaves* of $\mathcal{F}$) such that for each $p \in M$ there exists a chart (U, φ) around p and open polydisks $A \subset \mathbb{C}^{n-m}$ and $B \subset \mathbb{C}^m$ with the property that $\varphi : U \to A \times B$ maps the connected components of $\mathcal{L}_\alpha \cap U$ to the level sets $A \times \{w\}, w \in B$.

1.2. Singular Holomorphic Foliations by Curves. Now we study those foliations which are allowed to have some "tame" singularities. Recall that a subset E of a complex manifold M is called an *analytic subvariety* if each $p \in M$ has a neighborhood U on which there are holomorphic functions $f_j : U \to \mathbb{C}$, $1 \leq j \leq k$, such that $E \cap U = \{x \in U : f_j(x) = 0, 1 \leq j \leq k\}$. Evidently, every analytic subvariety of M is closed, hence $M \smallsetminus E$ is itself a complex manifold of the same dimension as M.

Definition 1.4. Let M be a complex manifold. A *singular holomorphic foliation by curves* $\mathcal{F}$ on M is a holomorphic foliation by curves on $M \smallsetminus E$, where E is an analytic subvariety of M of codimension > 1. A point $p \in E$ is called a *removable singularity* of $\mathcal{F}$ if there exists a chart (U, φ) around p, compatible with the atlas $\mathcal{A}$ of $\mathcal{F}$ restricted to $M \smallsetminus E$, in the sense that $\varphi \circ \varphi_i^{-1}$ and $\varphi_i \circ \varphi^{-1}$ have

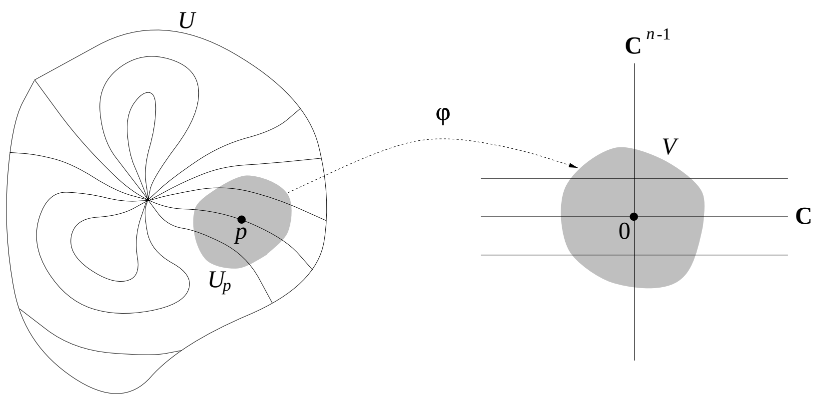

FIGURE 1. *Straightening a holomorphic vector field near a non-singular point.*

the form (1.1) for all $(U_i, \varphi_i) \in \mathcal{A}$ with $U \cap U_i \neq \emptyset$. The set of all non-removable singularities of $\mathcal{F}$ in E is called the *singular set* of $\mathcal{F}$, and is denoted by $\text{sing}(\mathcal{F})$.

Naturally, as in the case of real 1-dimensional singular foliations on real surfaces, the most important examples of singular holomorphic foliations are provided by vector fields.

EXAMPLE 1.5. Let $X = \sum_{i=1}^n f_i \; \partial/\partial z_i$ be a holomorphic vector field on a domain $U \subset \mathbb{C}^n$. We further assume that the Jacobian $(\partial f_i/\partial z_j)_{1 \leq i,j \leq n}$ has rank > 1 throughout the domain U. Then X vanishes on the analytic variety $\{z : f_1(z) = \cdots = f_n(z) = 0\}$ which has codimension > 1 (possibly the empty set). By definition, a *solution* of the differential equation $\dot{z} = X(z)$ with *initial condition* $p \in U$ is a holomorphic mapping $\eta : \mathbb{D}(0,r) \to U$ such that $\eta(0) = p$ and for every $T \in \mathbb{D}(0,r)$, $d\eta(T)/dT = X(\eta(T))$. The image $\eta(\mathbb{D}(0,r))$ is called a *local integral curve* passing through p. By the theorem of existence and uniqueness for the solutions of holomorphic differential equations [**IY**], each p has such a local integral curve passing through it, and two local integral curves through p coincide in some neighborhood of p. It follows that if $X(p) = 0$, then its integral curve will be the point p itself. If $X(p) \neq 0$, a local integral curve through p is a disk holomorphically embedded in U.

Now suppose that $X(p) \neq 0$. By the Straightening Theorem for holomorphic vector fields [**IY**], there exist neighborhoods $U_p \subset U$ of p and $V \subset \mathbb{C}^n$ of 0 and a biholomorphism $\varphi : U_p \to V$ such that $\varphi(p) = 0$ and $\varphi_*(X|_{U_p}) = \partial/\partial z_1$ (Fig. 1). Thus the connected components of the intersection of the local integral curves with U_p are mapped by φ to "horizontal" lines $\{z_2 = \text{const.}, \ldots, z_n = \text{const.}\}$. In other words, X induces a singular holomorphic foliation by curves on U, denoted by $\mathcal{F}_X$, with $\text{sing}(\mathcal{F}_X) = \{X = 0\}$, whose plaques are local integral curves of X.

Conversely, every singular holomorphic foliation by curves is locally induced by a holomorphic vector field (compare Proposition 1.11 below).

Here is one property of foliations which is quite elementary and will be frequently used in subsequent arguments:

Proposition 1.6. *Let $\mathcal{F}$ be a singular holomorphic foliation by curves on M. Suppose that $p \in \overline{\mathcal{L}_q}$. Then $\overline{\mathcal{L}_p} \subset \overline{\mathcal{L}_q}$.*

Proof. There is nothing to prove if $p \in \text{sing}(\mathcal{F})$, so let p be non-singular. Let $p' \in \mathcal{L}_p$ and join p to p' by a continuous path $\gamma : [0,1] \to \mathcal{L}_p$ (thus avoiding $\text{sing}(\mathcal{F})$) such that $\gamma(0) = p$ and $\gamma(1) = p'$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of [0,1] and foliation charts $(U_i, \varphi_i),\ 0 \leq i \leq n-1$, such that $\gamma[t_i, t_{i+1}] \subset U_i$. It follows from the local picture of plaques in $\varphi_i(U_i)$ that $\gamma(t_1) \in \overline{\mathcal{L}_q}$, so by repeating this argument, $p' \in \overline{\mathcal{L}_q}$. Since p' was arbitrary, we have $\overline{\mathcal{L}_p} \subset \overline{\mathcal{L}_q}$. □

Example 1.7. Example 1.5 has a counterpart in the context of differential forms defined on domains in $\mathbb{C}^n$. Our main example which will be seen to be quite general is the following. Let $\omega = P(x,y)dy - Q(x,y)dx$ be a holomorphic 1-form on $\mathbb{C}^2$, where P and Q are relatively prime polynomials. By definition, the singular foliation induced by ω, $\mathcal{F}_\omega : \{\omega = 0\}$, is the one induced on $\mathbb{C}^2$ by the vector field $X(x,y) = P(x,y)\partial/\partial x + Q(x,y)\partial/\partial y$ as in Example 1.5. In the language of 1-forms its leaves are obtained as follows: Take any $p \in \mathbb{C}^2$ at which P and Q are not simultaneously zero, and let $\eta : T \mapsto (x(T), y(T))$ be a holomorphic mapping on some disk $\mathbb{D}(0,r)$ which satisfies $\eta(0) = p$ and

$$P(x(T), y(T))\, y'(T) - Q(x(T), y(T))\, x'(T) = 0 \tag{1.2}$$

for all $T \in \mathbb{D}(0,r)$. The plaque through p is the image under η of some possibly smaller neighborhood of 0. Note that $\text{sing}(\mathcal{F}_\omega)$ is a finite set by Bezout's Theorem. Observe that

$$\mathcal{F}_\omega = \mathcal{F}_{f\omega} \tag{1.3}$$

for all holomorphic functions $f : \mathbb{C}^2 \to \mathbb{C}$ which are nowhere zero. It is exactly this property which allows us to extend the foliation from $\mathbb{C}^2$ to the complex projective plane $\mathbb{CP}^2$.

Convention 1.8. For the rest of the manuscript, the term "Singular Holomorphic Foliation by Curves" will be abbreviated as "SHFC".

Now we make a digression to study polynomial SHFC's on $\mathbb{CP}^2$ which are obtained by extending an $\mathcal{F}_\omega$ induced by a polynomial 1-form ω on $\mathbb{C}^2$.

1.3. Geometry of $\mathbb{CP}^2$. Consider $\mathbb{C}^3 \smallsetminus \{(0,0,0)\}$ with the action of $\mathbb{C}^*$ defined by $\lambda \cdot (x_0, x_1, x_2) = (\lambda x_0, \lambda x_1, \lambda x_2)$. The orbit of (x_0, x_1, x_2) is denoted by $[x_0, x_1, x_2]$. The quotient of $\mathbb{C}^3 \smallsetminus \{(0,0,0)\}$ modulo this action (with the quotient topology) is called the *complex projective plane* $\mathbb{CP}^2$, and the natural projection $\mathbb{C}^3 \smallsetminus \{(0,0,0)\} \to \mathbb{CP}^2$ is denoted by π. $\mathbb{CP}^2$ can be made into a compact complex 2-manifold in the following way: Cover $\mathbb{CP}^2$ by three open sets

$$U_i := \{[x_0, x_1, x_2] : x_i \neq 0\}, \qquad i = 0, 1, 2 \tag{1.4}$$

and define homeomorphisms $\phi_i : \mathbb{C}^2 \to U_i$ by

$$\begin{aligned} \phi_0(x,y) &= [1, x, y] \\ \phi_1(u,v) &= [u, 1, v] \\ \phi_2(r,s) &= [r, s, 1]. \end{aligned} \tag{1.5}$$

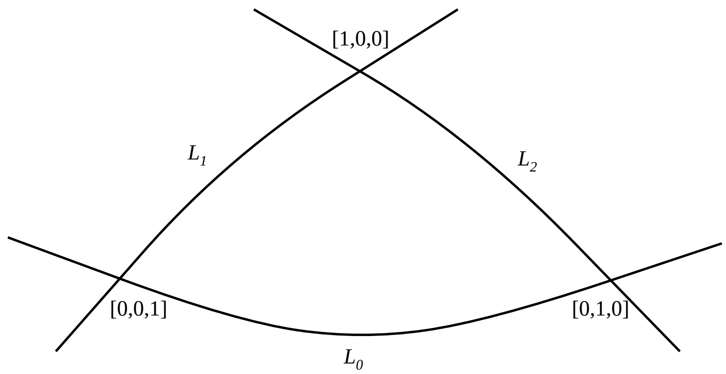

FIGURE 2. *Geometry of* $\mathbb{CP}^2$.

The change of coordinates $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ are given by

$$\phi_{01}(x,y) = \phi_1^{-1} \circ \phi_0(x,y) = \left(\frac{1}{x}, \frac{y}{x}\right)$$

$$\phi_{12}(u,v) = \phi_2^{-1} \circ \phi_1(u,v) = \left(\frac{u}{v}, \frac{1}{v}\right) \tag{1.6}$$

$$\phi_{20}(r,s) = \phi_0^{-1} \circ \phi_2(r,s) = \left(\frac{s}{r}, \frac{1}{r}\right).$$

These being holomorphic, the atlas $\{(U_i, \phi_i^{-1}), i = 0,1,2\}$ determines a unique complex structure on $\mathbb{CP}^2$ for which the ϕ_i are biholomorphisms. Intuitively, $\mathbb{CP}^2$ is a one-line compactification of $\mathbb{C}^2$ for the following reason: Each (U_i, ϕ_i^{-1}) is called an *affine chart* of $\mathbb{CP}^2$. Each $L_i := \mathbb{CP}^2 \smallsetminus U_i$ has a natural structure of the Riemann sphere; for example $L_0 = \{[0,x,y] : (x,y) \in \mathbb{C}^2\}$ can be identified with $\{[x,y] : (x,y) \in \mathbb{C}^2\} \simeq \mathbb{CP}^1$ under the restriction to $\mathbb{C}^2$ of the action of $\mathbb{C}^*$. Each L_i is called the *line at infinity* with respect to the affine chart (U_i, ϕ_i^{-1}). Fig. 2 shows the relative position of the lines L_i.

It is easy to check that given any *projective line* L in $\mathbb{CP}^2$, i.e., the projection under π of any plane $ax_0 + bx_1 + cx_2 = 0$ in $\mathbb{C}^3$, one can choose a biholomorphism $\phi : \mathbb{C}^2 \to \mathbb{CP}^2 \smallsetminus L$. In this way, L may be viewed as the line at infinity with respect to some affine chart.

1.4. Algebraic Curves in $\mathbb{CP}^2$. Suppose that $P = P(x,y) = \sum a_{ij}\ x^i y^j$ is a polynomial of degree k on $(x,y) \in \mathbb{C}^2$. Using (1.6), we write P in two other affine charts as

$$P \circ \phi_{10}(u,v) = P\left(\frac{1}{u}, \frac{v}{u}\right) = u^{-k} \sum a_{ij}\ u^{k-(i+j)} v^j$$

$$P \circ \phi_{20}(r,s) = P\left(\frac{s}{r}, \frac{1}{r}\right) = r^{-k} \sum a_{ij}\ r^{k-(i+j)} s^i.$$

Set $P'(u,v) = \sum a_{ij}\ u^{k-(i+j)} v^j$ and $P''(r,s) = \sum a_{ij}\ r^{k-(i+j)} s^i$. Then the *algebraic curve* S_P in $\mathbb{CP}^2$ is defined as the compact set

$$\phi_0\{(x,y) : P(x,y) = 0\} \cup \phi_1\{(u,v) : P'(u,v) = 0\} \cup \phi_2\{(r,s) : P''(r,s) = 0\}.$$

Another way of viewing this curve is by introducing the homogeneous polynomial $\widetilde{P}$ of degree k in $\mathbb{C}^3$ as

$$\widetilde{P}(x_0, x_1, x_2) := x_0^k\, P\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) = \sum a_{ij}\ x_1^i x_2^j x_0^{k-(i+j)}.$$

It is then easily verified that $S_P = \pi\{(x_0, x_1, x_2) : \widetilde{P}(x_0, x_1, x_2) = 0\}$. A projective line is an algebraic curve S_P for a polynomial P of degree 1.

1.5. Extending Polynomial 1-froms on $\mathbb{CP}^2$. Consider a polynomial 1-from $\omega = Pdy - Qdx$ on $\mathbb{C}^2$ and its corresponding SHFC $\mathcal{F}_\omega$, as in Example 1.7. Using the coordinate map ϕ_0 in (1.5) one can transport $\mathcal{F}_\omega$ to U_0. To complete this picture to all of $\mathbb{CP}^2$ we have to define the foliation on L_0. This can be done as follows. First transport $\mathcal{F}_\omega$ to the affine chart (u, v). To this end, write

$$\begin{aligned}
\tilde{\omega}(u,v) := \quad & (\phi_{10}^*\omega)(u,v) \\
= \quad & P\left(\frac{1}{u}, \frac{v}{u}\right) d\left(\frac{v}{u}\right) - Q\left(\frac{1}{u}, \frac{v}{u}\right) d\left(\frac{1}{u}\right) \\
= \quad & u^{-1}P\left(\frac{1}{u}, \frac{v}{u}\right) dv - u^{-2}\left\{vP\left(\frac{1}{u}, \frac{v}{u}\right) - Q\left(\frac{1}{u}, \frac{v}{u}\right)\right\} du.
\end{aligned}$$

Set $R(x, y) := yP(x, y) - xQ(x, y)$. Then

$$\tilde{\omega}(u,v) = u^{-1}P\left(\frac{1}{u}, \frac{v}{u}\right) dv - u^{-1}R\left(\frac{1}{u}, \frac{v}{u}\right) du.$$

Let k be the least positive integer such that $\omega' := u^{k+1}\tilde{\omega}$ is a polynomial 1-form on $(u, v) \in \mathbb{C}^2$. Two foliations $\mathcal{F}_{\omega'}$ and $\mathcal{F}_{\tilde{\omega}}$ are then identical on $\{(u, v) \in \mathbb{C}^2 : u \neq 0\}$ by (1.3); however, $\mathcal{F}_{\omega'}$ which is defined on all of $(u, v) \in \mathbb{C}^2$ is a well-defined extension of $\mathcal{F}_{\tilde{\omega}}$. Now transport $\mathcal{F}_{\omega'}$ by ϕ_1 to U_1. It is easily checked that the foliation induced by $(\mathcal{F}_\omega, \phi_0)$ coincides with that of $(\mathcal{F}_{\omega'}, \phi_1)$ on $U_0 \cap U_1$.

In a similar way, $\mathcal{F}_\omega$ can be transported to the affine chart (r, s) by ϕ_{20} to obtain a foliation $\mathcal{F}_{\omega''}$ induced by a polynomial 1-form ω'' on $(r, s) \in \mathbb{C}^2$. Then $\mathcal{F}_{\omega''}$ is transported to U_2 by ϕ_2.

We still denote the extended foliation on $\mathbb{CP}^2$ by $\mathcal{F}_\omega$ and frequently identify $\mathcal{F}_\omega, \mathcal{F}_{\omega'}, \mathcal{F}_{\omega''}$ with their transported companions on $\mathbb{CP}^2$. Thus, without saying explicitly, the affine charts $(x, y), (u, v)$, and (r, s) are considered as *subsets* of $\mathbb{CP}^2$ by identifying them with U_0, U_1 and U_2, respectively.

It follows from the above construction that in each affine chart, $\mathcal{F}_\omega$ is given by the integral curves of the following vector fields:

$$\begin{aligned}
&\text{In } (x,y) \in U_0, \quad X_0 = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \\
&\text{In } (u,v) \in U_1, \quad X_1 = u^k P\left(\frac{1}{u}, \frac{v}{u}\right)\frac{\partial}{\partial u} + u^k R\left(\frac{1}{u}, \frac{v}{u}\right)\frac{\partial}{\partial v} \qquad (1.7) \\
&\text{In } (r,s) \in U_2, \quad X_2 = -r^l Q\left(\frac{s}{r}, \frac{1}{r}\right)\frac{\partial}{\partial r} + r^l R\left(\frac{s}{r}, \frac{1}{r}\right)\frac{\partial}{\partial s},
\end{aligned}$$

where k and l are the least positive integers making the above into polynomial vector fields. This shows that $\mathcal{F}_\omega$ is an SHFC on $\mathbb{CP}^2$ with $\text{sing}(\mathcal{F}_\omega) = S_P \cap S_Q \cap S_R$.

1.6. Holomorphic Vector Fields on $\mathbb{CP}^2$. So far it should seem to be that leaves of the SHFC's $\mathcal{F}_\omega$ constructed on $\mathbb{CP}^2$ are realized as integral curves of holomorphic *line fields* on $\mathbb{CP}^2$ rather than *vector fields.* This distinction should have been observed when we multiplied $\tilde{\omega}$ by a power of u to cancel the pole at $u = 0$ (compare §1.5). In fact, the special geometry of the projective plane puts a severe restriction on holomorphic vector fields on $\mathbb{CP}^2$. As a result, very few SHFC's on $\mathbb{CP}^2$ can be globally described by a single holomorphic vector field.

The following proposition describes all holomorphic vector fields on $\mathbb{CP}^2$. The same characterization is true for $\mathbb{CP}^n$, as can be shown by a coordinate-free argument [**CKP**], but here we present a very elementary proof for $\mathbb{CP}^2$.

Proposition 1.9. *Every holomorphic vector field on $\mathbb{CP}^2$ lifts to a linear vector field on $\mathbb{C}^3$.*

Proof. Let X be a holomorphic vector field on $\mathbb{CP}^2$, which has the following expressions in the three affine charts U_0, U_1, and U_2:

$$\text{In } (x,y) \in U_0 \ : \ X_0 = f_0 \, \frac{\partial}{\partial x} + g_0 \, \frac{\partial}{\partial y}$$

$$\text{In } (u,v) \in U_1 \ : \ X_1 = f_1 \, \frac{\partial}{\partial u} + g_1 \, \frac{\partial}{\partial v}$$

$$\text{In } (r,s) \in U_2 \ : \ X_2 = f_2 \, \frac{\partial}{\partial r} + g_2 \, \frac{\partial}{\partial s}.$$

Since $(\phi_{01})_* X_0 = X_1$ and $(\phi_{02})_* X_0 = X_2$, we obtain

$$f_1(u,v) = -u^2 f_0 \left(\frac{1}{u}, \frac{v}{u} \right) \tag{1.8}$$

$$g_1(u,v) = -uv \ f_0 \left(\frac{1}{u}, \frac{v}{u} \right) + u \ g_0 \left(\frac{1}{u}, \frac{v}{u} \right), \tag{1.9}$$

and

$$f_2(r,s) = -r^2 g_0 \left(\frac{s}{r}, \frac{1}{r} \right) \tag{1.10}$$

$$g_2(r,s) = r f_0 \left(\frac{s}{r}, \frac{1}{r} \right) - rs \ g_0 \left(\frac{s}{r}, \frac{1}{r} \right). \tag{1.11}$$

Consider the power series expansions

$$f_0(x,y) = \sum_{i,j \geq 0} a_{ij} \ x^i y^j \quad \text{and} \quad g_0(x,y) = \sum_{i,j \geq 0} b_{ij} \ x^i y^j.$$

Since f_1, g_1, f_2, g_2 are holomorphic on $\mathbb{C}^2$, we have the following:

$$\begin{array}{llll} a_{ij} = 0 & \text{if} & i+j \geq 3 & (\text{by}(1.8)) \\ b_{ij} = 0 & \text{if} & i+j \geq 3 & (\text{by}(1.9)) \\ a_{02} = b_{20} = 0 & a_{20} = b_{11} & b_{02} = a_{11} & (\text{by}(1.9)) \end{array}$$

and (1.10) and (1.11) give no new relations. Now it can be easily checked that X lifts to the linear vector field

$$\begin{aligned}\tilde{X} &= (x_0 - a_{20}x_1 - a_{11}x_2)\frac{\partial}{\partial x_0} \\ &+(a_{00}x_0 + (a_{10}+1)x_1 + a_{01}x_2)\frac{\partial}{\partial x_1} \\ &+(b_{00}x_0 + b_{10}x_1 + (b_{01}+1)x_2)\frac{\partial}{\partial x_2}\end{aligned}$$

on $\mathbb{C}^3$. Conversely, every linear vector field on $\mathbb{C}^3$ descends to a holomorphic vector field on $\mathbb{CP}^2$, and we are done. □

As a result, every holomorphic vector field on $\mathbb{CP}^2$ is seen in the affine chart (x, y) as $X_0 = f_0\ \partial/\partial x + g_0\ \partial/\partial y$, where

$$\begin{aligned} f_0(x,y) &= a_1 + a_2x + a_3y + x(Ax+By) \\ g_0(x,y) &= b_1 + b_2x + b_3y + y(Ax+By), \end{aligned} \tag{1.12}$$

for some complex constants a_i, b_i, A, B.

Our next goal is to give in detail the proof of the remarkable fact that every SHFC on $\mathbb{CP}^2$ is of the form $\mathcal{F}_\omega$ for some polynomial 1-form ω (equivalently, a polynomial vector field) on $\mathbb{C}^2$. The same proof works for every $\mathbb{CP}^n$, $n \geq 2$, with only minor modifications. We refer the reader to [**GO**] for a more general set up. Another proof for this fact can be given by methods of algebraic geometry (see [**I3**]).

The proof goes along the following lines: First we associate to each SHFC on a complex manifold M a (holomorphic) line bundle $B' \hookrightarrow TM$ over $M \smallsetminus E$ (E is an analytic subvariety of M, as in Definition 1.3). Then we show that B' may be extended to a tangent line bundle B over M (Theorem 1.12). This establishes a natural relationship between SHFC's on M and holomorphic "bundle maps" $\beta : B \to TM$. Rigidity of line bundles in the case $M = \mathbb{CP}^2$ will then be applied to show that each bundle map $\beta : B \to T\mathbb{CP}^2$ is induced by a polynomial 1-form on $\mathbb{C}^2$ (Corollary 1.17). The foundational material on holomorphic line bundles on complex manifolds used here can be found in [**GH**] or [**Kod**].

1.7. SHFC's and Line Bundles. Let $\mathcal{F}$ be an SHFC on a complex manifold M. Let $\{(U_i, \varphi_i)\}_{i\in I}$ be the collection of foliation charts on $M' := M \smallsetminus E$. By (1.1) the transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} = (\varphi_{ij}^1, \ldots, \varphi_{ij}^n)$ satisfy $\partial\varphi_{ij}^k/\partial z_1 = 0$ for $2 \leq k \leq n$. Applying the chain rule to $\varphi_{ij}^1 = \varphi_{ik}^1 \circ \varphi_{kj}^1$, we obtain

$$\frac{\partial\varphi_{ij}^1}{\partial z_1}(p) = \frac{\partial\varphi_{ik}^1}{\partial z_1}(\varphi_{kj}(p))\frac{\partial\varphi_{kj}^1}{\partial z_1}(p) \tag{1.13}$$

for every $p \in \varphi_j(U_i \cap U_j \cap U_k)$. Define $\xi_{ij} : U_i \cap U_j \to \mathbb{C}^*$ by

$$\xi_{ij}(p) := \frac{\partial\varphi_{ij}^1}{\partial z_1}(\varphi_j(p)). \tag{1.14}$$

The cocycle relation $\xi_{ij} = \xi_{ik} \cdot \xi_{kj}$ on $U_i \cap U_j \cap U_k$ follows from (1.13). Let B' be the holomorphic line bundle on M' defined by this cocycle. This line bundle has a natural holomorphic injection into TM in such a way that the image of the fiber over p under this injection coincides with the tangent line to $\mathcal{L}_p$ at p. To see this, observe that

$$B' = \bigcup_{i \in I} (U_i \times \mathbb{C}) / \sim,$$

where $(p, t) \in U_i \times \mathbb{C}$ is identified under $\sim$ with $(q, t') \in U_j \times \mathbb{C}$ if and only if $p = q$ and $t = \xi_{ij}(p)t'$. For $p \in M'$, let B'_p be the fiber of B' over p and define $\beta'_i : U_i \times \mathbb{C} \to TM|_{U_i}$ by

$$\beta'_i(p, t) := t((\varphi_i^{-1})_* \frac{\partial}{\partial z_1})(p). \tag{1.15}$$

It follows then from (1.14) that if $p \in U_i \cap U_j$ and $(p, t) \sim (p, t')$, then $\beta'_i(p, t) = \beta'_j(p, t')$. Therefore, the β'_i give rise to a holomorphic bundle map $\beta' : B' \to TM$ which is injective and $\beta'(B'_p)$ is the tangent line to $\mathcal{L}_p$ at p, a subspace of T_pM.

Can B' be extended to a line bundle over all of M? The crucial point for the answer, which is affirmative, is the condition that the codimension of E is > 1. Recall the following classical theorem of F. Hartogs (see for example [**W**] for a proof):

Theorem 1.10. *Let $U \subset \mathbb{C}^n$ be a domain and $E \subset U$ be an analytic subvariety of U of codimension > 1. Then every holomorphic (resp. meromorphic) function on $U \smallsetminus E$ can be extended to a holomorphic (resp. meromorphic) function on U.*

Using this theorem one can extend line bundles induced by SHFC's. To this end, we have to prove the following simple but remarkable proposition (compare [**GO**]). The letter E will denote an analytic subvariety of the ambient space which satisfies Definition 1.4.

Proposition 1.11. *Let $U \subset \mathbb{C}^n$ be a domain and $\mathcal{F}$ be an SHFC on U. Then for each $p \in U$ there exists a holomorphic vector field X on some neighborhood U_p of p such that X is non-vanishing on $U_p \smallsetminus E$ and is tangent to the leaves of $\mathcal{F}$. This X is unique up to multiplication by a holomorphic function which is non-zero in a neighborhood of p. Moreover, $q \in U_p \cap E$ is a removable singularity of $\mathcal{F}$ if and only if $X(q) \neq 0$.*

Proof. There is nothing to prove if $p \notin E$, so let $p \in E$ and let U_p be a connected neighborhood of p in U. Then $U'_p := U_p \smallsetminus E$ is open and connected. Each $q \in U'_p$ has a small connected neighborhood $U_q \subset U'_p$ on which there is a holomorphic vector field representing $\mathcal{F}$ on U_q. Without loss of generality we may assume that the first component of these vector fields is not identically zero over U'_p. If $(Y_1, \ldots, Y_n)$ is the vector field representing $\mathcal{F}$ on U_q, then $(1, Y_2/Y_1, \ldots, Y_n/Y_1)$ is a meromorphic vector field on U_q representing $\mathcal{F}$ on $U_q \smallsetminus \{\text{zeros of } Y_1\}$. Repeating this argument for each $q \in U'_p$, and noting that away from E the vector field representing $\mathcal{F}$ is uniquely determined up to multiplication by a non-vanishing holomorphic function, one concludes that there are meromorphic functions $F_2, \ldots, F_n$ defined on U'_p such that $(1, F_2, \ldots, F_n)$ represents $\mathcal{F}$ on $U'_p \smallsetminus \{\text{poles of the } F_i \text{ in } U'_p\}$. By Theorem 1.10, each function F_i can be extended to a meromorphic function on U_p (still denoted by F_i) since the codimension of $U_p \cap E$ is > 1. The germ of F_i can be uniquely written as $F_i = f_i/g_i$ by choosing U_p small enough, where f_i and g_i are relatively prime holomorphic functions on U_p. Then $X := (g, gF_2, \ldots, gF_n)$ is a holomorphic vector field on U_p representing $\mathcal{F}$ on $U'_p \smallsetminus \{\text{zeros of } g\}$, where g

is the least common multiple of the g_i. Note that codim $\{z : X(z) = 0\} > 1$ (the components of X are relatively prime).

Now let $q \in U'_p$, and (U, φ) be a foliation chart around q. The vector field $\varphi_* X$ has the form $h\, \partial/\partial z_1$ since it is tangent to the horizontal lines $\{z_2 = \text{const.}, \ldots, z_n = \text{const.}\}$ away from zeros of $g \circ \varphi^{-1}$. Since the zero set of h either is empty or has codimension 1, while the zero set of $\varphi_* X$ has codimension >1, it follows that h is nowhere zero and $X(q) \neq 0$. Thus $X(q)$ is tangent to $\mathcal{L}_q$ at q.

For the uniqueness part, let $\tilde{X}$ be another such vector field. Then $\tilde{X} = \xi X$ on U'_p, where $\xi : U'_p \to \mathbb{C}^*$ is holomorphic. Extend ξ over U_p by Theorem 1.10. This new ξ is nowhere vanishing, since its zero set, if non-empty, would have codimension 1.

Finally, let $q \in U_p \cap E$ be a removable singularity of $\mathcal{F}$. Choose a compatible chart (U, φ) around q (compare Definition 1.4) and let $\tilde{X} = \varphi_*^{-1}(\partial/\partial z_1)$. Then $\tilde{X}$ describes $\mathcal{F}$ on U, so by the above uniqueness we have $\tilde{X} = \xi X$, with ξ being a holomorphic non-vanishing function on some neighborhood of q. Thus $X(q) \neq 0$. Conversely, suppose that $X(q) \neq 0$, and let (U, φ) be a local chart around q straightening the integral curves of X, i.e., $\varphi_*(X|_U) = \partial/\partial z_1$. Then it is evident that (U, φ) is compatible with every foliation chart of $\mathcal{F}$ in U'_p. □

Now let $\mathcal{F}$ be an SHFC on M and $\beta' : B' \to TM$ be the bundle map constructed in (1.15). According to Proposition 1.11, each $p \in M$ has a neighborhood U_i and a holomorphic vector field X_i defined on U_i representing $\mathcal{F}$ on $U_i \smallsetminus E$. By the uniqueness part of Proposition 1.11, whenever $U_i \cap U_j \neq \emptyset$, we have $X_j = \xi_{ij} X_i$ on $U_i \cap U_j$, where $\xi_{ij} : U_i \cap U_j \to \mathbb{C}^*$ is holomorphic. Let B be the line bundle over M defined by the cocycle $\{\xi_{ij}\}$. As in the construction of β' in (1.15), define $\beta_i : U_i \times \mathbb{C} \to TM|_{U_i}$ by

$$\beta_i(p, t) := t X_i(p). \tag{1.16}$$

The definition of B shows that the β_i patch together to yield a well-defined bundle map $\beta : B \to TM$ for which $\beta(B_p)$ is the tangent line to $\mathcal{L}_p$ at p if $p \notin E$. Note that B is an extension of B' since $\beta^{-1} \circ \beta' : B' \to B|_{M'}$ is an isomorphism of line bundles. If $\eta : B \to TM$ is another bundle map which represents $\mathcal{F}$ away from E, then $\eta : U_i \times \mathbb{C} \to TM|_{U_i}$ satisfies $\eta(p, t) = t\lambda_i(p) X_i(p)$ for $p \in U_i \smallsetminus E$, where $\lambda_i : U_i \smallsetminus E \to \mathbb{C}$ is non-vanishing. Extend λ_i to U_i by Theorem 1.10. Note that the action of η on B_p is well-defined, so that $\lambda_i(p) = \lambda_j(p)$ if $p \in U_i \cap U_j$. Defining $\lambda : M \to \mathbb{C}$ by $\lambda|_{U_i} := \lambda_i$, we see that $\eta = \lambda \cdot \beta$. Furthermore, λ can only vanish on E, but its zero set, if non-empty, would have to have codimension 1. Therefore, λ does not vanish at all. Finally, the last part of Proposition 1.11 and (1.16) show that $p \in E$ is a removable singularity of $\mathcal{F}$ if and only if $\beta(B_p) \neq 0$.

Summarizing the above argument, we obtain

THEOREM 1.12. *The line bundle B' over $M \smallsetminus E$ associated to an SHFC $\mathcal{F}$ on M can be extended to a line bundle B over M. There exists a holomorphic bundle map $\beta : B \to TM$ for which $\beta(B_p)$ is the tangent line to $\mathcal{L}_p$ at p if $p \notin E$. This β is unique up to multiplication by a nowhere vanishing holomorphic function on M. A point $p \in E$ is a removable singularity of $\mathcal{F}$ if and only if $\beta(B_p) \neq 0$.*

REMARK 1.13. It follows from the last part of Theorem 1.12 that sing$(\mathcal{F})$ is precisely $\{p \in M : \beta(B_p) = 0\}$; in particular, after removing all the possible removable singularities of $\mathcal{F}$ in E, $\mathrm{sing}(\mathcal{F})$ turns out to be an analytic subvariety of M of codimension > 1.

Remark 1.14. If M is a *compact* complex manifold (in particular, if $M = \mathbb{CP}^2$), then every two bundle maps $\beta, \beta' : B \to TM$ representing $\mathcal{F}$ differ by a non-zero multiplicative constant.

Remark 1.15. The bundle map constructed in the above theorem is unique in the sense that *two bundle maps $\beta : B \to TM$ and $\tilde{\beta} : \tilde{B} \to TM$ represent the same SHFC if and only if there exists a bundle isomorphism $\psi : B \to \tilde{B}$ with $\beta = \tilde{\beta} \circ \psi$.* To see this, first assume that both bundle maps represent $\mathcal{F}$. By the construction, we may assume that B (resp. $\tilde{B}$) is defined by $(\{U_i\}, \{\xi_{ij}\})$ (resp. $(\{V_k\}, \{\eta_{kl}\})$) and there are holomorphic vector fields X_i on U_i (resp. Y_k on V_k) satisfying $X_j = \xi_{ij} X_i$ on $U_i \cap U_j$ (resp. $Y_l = \eta_{kl} Y_k$ on $V_k \cap V_l$) and $\beta_i(p,t) := tX_i(p)$ (resp. $\tilde{\beta}_k(p,t) := tY_k(p)$). Since B and $\tilde{B}$ both represent $\mathcal{F}$, for every i, k with $U_i \cap V_k \neq \emptyset$, there is a nowhere vanishing holomorphic function λ_{ik} such that $X_i(p) = \lambda_{ik}(p) Y_k(p)$ for all $p \in (U_i \cap V_k) \smallsetminus \mathrm{sing}(\mathcal{F})$. By Theorem 1.10, λ_{ik} can be extended to $U_i \cap V_k$. Note that the extended function cannot vanish at all, since its only possible zero set is $U_i \cap V_k \cap \mathrm{sing}(\mathcal{F})$ which has codimension >1. Now define $\psi : B \to \tilde{B}$ by mapping the class of $(p,t) \in U_i \times \mathbb{C}$ to the class of $(p, t\lambda_{ik}(p)) \in V_k \times \mathbb{C}$. It is quite easy to see that ψ defines an isomorphism of line bundles with $\beta = \tilde{\beta} \circ \psi$.

Conversely, let $\beta : B \to TM$ represent $\mathcal{F}$. If $\tilde{B}$ is any line bundle over M isomorphic to B by $\psi : B \to \tilde{B}$, and if $\tilde{\beta} = \beta \circ \psi^{-1}$, then $\tilde{\beta} : \tilde{B} \to TM$ also represents $\mathcal{F}$.

1.8. Line Bundles and $H^1(M, \mathcal{O}^*)$. Every holomorphic line bundle B on a complex manifold M is uniquely determined by an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of M and a family $\{\xi_{ij}\}_{i,j \in I}$ of non-vanishing holomorphic functions on each $U_i \cap U_j \neq \emptyset$ satisfying the cocycle relation $\xi_{ij} = \xi_{ik} \cdot \xi_{kj}$ on $U_i \cap U_j \cap U_k$. Thus $\{\xi_{ij}\}$ may be regarded as an element of $Z^1(\mathcal{U}, \mathcal{O}^*)$, the group of Čech 1-cocycles with coefficients in the sheaf of non-vanishing holomorphic functions on M, with respect to the covering $\mathcal{U}$. Let B' be another line bundle on M defined by cocycle $\{\eta_{ij}\}$. It is not difficult to see that B' is isomorphic to B if and only if there exist non-vanishing holomorphic functions $f_i : U_i \to \mathbb{C}^*$ such that $\eta_{ij} = (f_i/f_j)\xi_{ij}$ on $U_i \cap U_j$. Interpreting $\{\xi_{ij}\}$ and $\{\eta_{ij}\}$ as elements of $Z^1(\mathcal{U}, \mathcal{O}^*)$, the last condition may be written as $\{\xi_{ij}/\eta_{ij}\} \in B^1(\mathcal{U}, \mathcal{O}^*)$, the group of Čech 1-coboundaries. We conclude that *two holomorphic line bundles on M are isomorphic if and only if they represent the same element of the Čech cohomology group $H^1(M, \mathcal{O}^*) := Z^1(M, \mathcal{O}^*)/B^1(M, \mathcal{O}^*)$.*

1.9. Line Bundles on $\mathbb{CP}^2$. Most of the results presented here are true for $\mathbb{CP}^n$, $n \geq 2$. However, we only treat the case $n = 2$ for simplicity of exposition.

Consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

on M. From this sequence we obtain the long exact sequence of cohomology groups

$$\cdots \to H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O}) \to \cdots \tag{1.17}$$

For every line bundle $B \in H^1(M, \mathcal{O}^*)$, $c_1(B) \in H^2(M, \mathbb{Z})$ is called the *first Chern class* of B. Explicitly, let $B = [\{\xi_{ij}\}]$ be defined for a covering $\mathcal{U} = \{U_i\}$ for which every U_i is connected, every $U_i \cap U_j$ is simply-connected, and every $U_i \cap U_j \cap U_k$

is connected. Then $c_1(B) = [\{c_{ijk}\}]$, where $c_{ijk} := 1/(2\pi\sqrt{-1})\{\log\xi_{jk} - \log\xi_{ik} + \log\xi_{ij}\}$ and the branches of logarithms are arbitrarily chosen [**Kod**].

There are two basic facts about the sequence (1.17) in the case $M = \mathbb{CP}^2$. First, for every $i \geq 1$ we have $H^i(\mathbb{CP}^2, \mathcal{O}) = 0$ as a consequence of the Hodge Decomposition Theorem [**GH**]. Second, $H^2(\mathbb{CP}^2, \mathbb{Z}) \simeq \mathbb{Z}$ [**GH**]. It follows that c_1 in (1.17) is an isomorphism, $H^1(\mathbb{CP}^2, \mathcal{O}^*)$ is the infinite cyclic group $\mathbb{Z}$, and every holomorphic line bundle on $\mathbb{CP}^2$ is determined up to isomorphism by its first Chern class (compare §1.8).

Now consider $\mathbb{CP}^2$ with affine charts $\{(U_i, \phi_i^{-1})\}_{i=0,1,2}$, as in §1.3. For every integer n, define a line bundle $B(n)$ on $\mathbb{CP}^2$ whose cocycle $\{\xi_{ij}\}_{i,j=0,1,2}$ is given by

$$\xi_{ij} : U_i \cap U_j \to \mathbb{C}^*, \quad i, j = 0, 1, 2$$

$$\xi_{ij}[x_0, x_1, x_2] := \left(\frac{x_j}{x_i}\right)^n. \tag{1.18}$$

It is not difficult to check that $B(-1)$, called the *canonical line bundle* over $\mathbb{CP}^2$, and its dual bundle $B(1)$ are both generators for the infinite cyclic group $H^1(\mathbb{CP}^2, \mathcal{O}^*)$. By a standard convention we define $c_1(B(1)) = 1$. It follows that $c_1(B(n)) = n$ for all integers n.

Adding up the above remarks, it follows that *every holomorphic line bundle B on $\mathbb{CP}^2$ is isomorphic to $B(n)$, where $n = c_1(B)$.*

Now suppose that $\mathcal{F}$ is an SHFC on $\mathbb{CP}^2$, and let $\beta : B \to T\mathbb{CP}^2$ be its associated bundle map given by Theorem 1.12. Choose $n = c_1(B)$ so that B is isomorphic to $B(n)$ by some $\psi : B \to B(n)$. By Remark 1.15, if $\tilde{\beta}$ is defined by $\tilde{\beta} := \beta \circ \psi^{-1}$, then $\tilde{\beta} : B(n) \to T\mathbb{CP}^2$ is a bundle map which also represents $\mathcal{F}$.

1.10. Explicit Form of SHFC's on $\mathbb{CP}^2$. In view of the remarks in the last paragraph, we are now going to determine the explicit form of every holomorphic bundle map $\beta : B(-n+1) \to T\mathbb{CP}^2$ for every integer n. (We choose $-n+1$ instead of n just to make later formulations easier.)

Once again consider $\mathbb{CP}^2$ equipped with the three affine charts $(U_i, \phi_i^{-1}), i = 0, 1, 2$. Restricting β to $U_i \times \mathbb{C}$, there exists a holomorphic vector field X_i on U_i representing the action of $\beta_i : U_i \times \mathbb{C} \to TU_i \simeq U_i \times \mathbb{C}^2$. In other words, $\beta_i(p, t) = (p, tX_i(p))$. Let $X_0 := f(x, y)\partial/\partial x + g(x, y)\partial/\partial y$ and $X_1 := \tilde{f}(u, v)\partial/\partial u + \tilde{g}(u, v)\partial/\partial v$, where $f, g, \tilde{f}, \tilde{g}$ are holomorphic on $\mathbb{C}^2$. Set $\tilde{U}_0 := U_0 \smallsetminus \{(x, y) : x = 0\}$ and $\tilde{U}_1 := U_1 \smallsetminus \{(u, v) : u = 0\}$. We have the following commutative diagram:

$$\begin{array}{ccc}
\tilde{U}_0 \times \mathbb{C} & \xrightarrow{\xi} & \tilde{U}_1 \times \mathbb{C} \\
\Big\downarrow \beta_0 & & \Big\downarrow \beta_1 \\
T\tilde{U}_0 \simeq \tilde{U}_0 \times \mathbb{C}^2 & \xrightarrow{(\phi_{01})_*} & T\tilde{U}_1 \simeq \tilde{U}_1 \times \mathbb{C}^2
\end{array}$$

where $\xi(x, y, t) := (u, v, t/\xi_{01}(x, y))$. Note that by (1.18), $\xi_{01}(x, y) = \xi_{01}[1, x, y] = x^{-n+1} = u^{n-1}$ so that $\xi(x, y, t) = (u, v, u^{-n+1}t)$. It follows that $(\phi_{01})_* X_0 =$

$u^{-n+1}X_1$, or

$$\tilde{f}(u,v) = -u^{n+1} f\left(\frac{1}{u}, \frac{v}{u}\right)$$
$$\tilde{g}(u,v) = -u^{n}\left\{ v f\left(\frac{1}{u}, \frac{v}{u}\right) - g\left(\frac{1}{u}, \frac{v}{u}\right)\right\}. \tag{1.19}$$

Since $\tilde{f}$ and $\tilde{g}$ are holomorphic on $\mathbb{C}^2$, (1.19) shows that if $n \leq -1$, then f and g both vanish. So let us assume that $n \geq 0$.

Suppose that $f = \sum_{k=0}^{\infty} f_k$ and $g = \sum_{k=0}^{\infty} g_k$, where f_k and g_k are the homogeneous parts of degree k of the power series expansions of f and g. It follows then from (1.19) that $f_k \equiv g_k \equiv 0$ for $k \geq n+2$, and

$$v f_{n+1}(1,v) - g_{n+1}(1,v) \equiv 0. \tag{1.20}$$

Coming back to the affine chart $(x,y) \in U_0$, we obtain from (1.20) that

$$y f_{n+1}(x,y) - x g_{n+1}(x,y) \equiv 0. \tag{1.21}$$

It is easy to see that there are no other restrictions on these homogeneous polynomials. Thus we have proved the following

THEOREM 1.16. *Let $\beta : B(-n+1) \to T\mathbb{CP}^2$ be a holomorphic bundle map. If $n \leq -1$, then $\beta \equiv 0$. If $n \geq 0$, then in the affine chart $(x,y) \in U_0$, β is given by a polynomial vector field of the form*

$$\sum_{k=0}^{n+1} f_k \frac{\partial}{\partial x} + \sum_{k=0}^{n+1} g_k \frac{\partial}{\partial y},$$

where f_k and g_k are homogeneous polynomials of degree k, and $y f_{n+1} - x g_{n+1} \equiv 0$.

Now let $\mathcal{F}$ be an SHFC on $\mathbb{CP}^2$ and let $\beta : B \to T\mathbb{CP}^2$ represent $\mathcal{F}$. Set $n = -c_1(B) + 1$. Then, by the above theorem $\mathcal{F}$ is induced by a polynomial 1-form

$$\omega = [(\sum_{k=0}^{n} f_k) + xh]\, dy - [(\sum_{k=0}^{n} g_k) + yh]\, dx,$$

in the affine chart U_0, where $h = f_{n+1}/x = g_{n+1}/y$ is a homogeneous polynomial of degree n or $h \equiv 0$. Let us assume for a moment that the latter happens, i.e., $f_{n+1} \equiv g_{n+1} \equiv 0$. Then, rewriting (1.19) gives us

$$\tilde{f}(u,v) = -u^{n+1} \sum_{k=0}^{n} u^{-k} f_k(1,v)$$
$$\tilde{g}(u,v) = -u^{n} \sum_{k=0}^{n} u^{-k} [v f_k(1,v) - g_k(1,v)]. \tag{1.22}$$

This shows that $v f_n(1,v) - g_n(1,v) \not\equiv 0$ since otherwise $\tilde{f}$ and $\tilde{g}$ would have a common factor u, meaning that β would vanish on the entire line $u = 0$ (contradicting the fact that β must vanish at finitely many points). We summarize the above observations in the following

Corollary 1.17. *Let $\mathcal{F}$ be an SHFC on $\mathbb{CP}^2$ and $\beta : B \to T\mathbb{CP}^2$ be any holomorphic bundle map representing $\mathcal{F}$. Then $c_1(B) \leq 1$. If $n = -c_1(B)+1$, then $\mathcal{F}$ is given by a unique (up to a multiplicative constant) polynomial 1-form*

$$\omega = (f + xh)\, dy - (g + yh)\, dx \tag{1.23}$$

in the affine chart $(x, y) \in U_0$. Here

- *$f = \sum_{k=0}^{n} f_k$ and $g = \sum_{k=0}^{n} g_k$, with f_k and g_k being homogeneous polynomials of degree k,*
- *either h is a non-zero homogeneous polynomial of degree n, or if $h \equiv 0$, then $yf_n - xg_n \not\equiv 0$,*
- *the two polynomials $f + xh$ and $g + yh$ have no common factor.*

1.11. Geometric Degree of an SHFC on $\mathbb{CP}^2$. Here we show that the "cohomological degree" $n = -c_1(B) + 1$ of an SHFC $\mathcal{F}$ given by Corollary 1.17 coincides with a geometric invariant which we will call the "geometric degree" of $\mathcal{F}$. Roughly speaking, the geometric degree of $\mathcal{F}$ is the number of points at which a generic projective line is tangent to the leaves of $\mathcal{F}$. This notion allows us to give a stratification of the space of all SHFC's on $\mathbb{CP}^2$.

Let $\mathcal{F}$ be an SHFC on $\mathbb{CP}^2$ and L be any projective line such that $L \smallsetminus \operatorname{sing}(\mathcal{F})$ is not a leaf of $\mathcal{F}$. A point $p \in L$ is called a *tangency point* of $\mathcal{F}$ and L if either $p \in \operatorname{sing}(\mathcal{F})$, or $p \notin \operatorname{sing}(\mathcal{F})$ and L is the tangent line to $\mathcal{L}_p$ at p.

Let $\mathcal{F}$ be of the form $\mathcal{F}_\omega : \{\omega = Pdy - Qdx = 0\}$ in the affine chart $(x, y) \in U_0$, and $L \cap U_0$ be parametrized by $\ell(T) = (x_0 + aT, y_0 + bT)$, where $p = (x_0, y_0) = \ell(0)$. Then it is clear that p is a tangency point of $\mathcal{F}$ and L if and only if $T = 0$ is a root of the polynomial $T \mapsto bP(\ell(T)) - aQ(\ell(T))$. The *order* of tangency of $\mathcal{F}$ and L at p is defined to be the multiplicity of $T = 0$ as a root of this polynomial. Define

$$m(\mathcal{F}, L) := \sum_p (\text{order of tangency of } \mathcal{F} \text{ and } L \text{ at } p),$$

where the (finite) sum is taken over all the tangency points.

In the theorem below, we show that $m(\mathcal{F}, L)$ does not depend on L (as long as $L \smallsetminus \operatorname{sing}(\mathcal{F})$ is not a leaf), so that we can call it the *geometric degree* of $\mathcal{F}$.

Theorem 1.18. *Let $\mathcal{F}$ be an SHFC on $\mathbb{CP}^2$ and $n = -c_1(B) + 1$, where B is the line bundle associated with $\mathcal{F}$. Let L be any projective line such that $L \smallsetminus \operatorname{sing}(\mathcal{F})$ is not a leaf. Then $m(\mathcal{F}, L) = n$. In particular, an SHFC $\mathcal{F}$ has geometric degree n if and only if $\mathcal{F}$ is induced by a 1-form ω as in (1.23) in which h is a non-zero homogeneous polynomial of degree n, or $h \equiv 0$ and $yf_n - xg_n \not\equiv 0$.*

Proof. Let $n = -c_1(B) + 1$ so that $\mathcal{F}$ is induced by a 1-form ω as in (1.23). Take a projective line L such that $L \smallsetminus \operatorname{sing}(\mathcal{F})$ is not a leaf. Since the normal form (1.23) is invariant under projective transformations, we may assume that L is the x-axis. By (1.23), $(x, 0) \in L \cap U_0$ is a tangency point if and only if $g(x, 0) = \sum_{k=0}^{n} g_k(x, 0) = 0$. As for the point at infinity for L, consider the affine chart $(u, v) = (1/x, y/x) \in U_1$ in which L is given by the line $\{v = 0\}$. By (1.19), the foliation is described by the polynomial 1-form $\omega' = \tilde{f}\, dv - \tilde{g}\, du$, where

$$\tilde{f}(u,v) = -\sum_{k=0}^{n} u^{n+1-k} f_k(1,v) - h(1,v)$$
$$\tilde{g}(u,v) = -\sum_{k=0}^{n} u^{n-k}[v f_k(1,v) - g_k(1,v)]. \tag{1.24}$$

This shows that L has a tangency point at infinity if and only if $u = 0$ is a root of $\tilde{g}(u,0) = 0$, or $g_n(1,0) = 0$ by (1.24). To prove the theorem, we distinguish two cases:

(i) The polynomial $x \mapsto g_n(x,0)$ is not identically zero. In this case, $\sum_{k=0}^{n} g_k(x,0)$ is a polynomial of degree n in x, so there are exactly n finite tangency points on L counting multiplicities. Note that the point at infinity for L is not a tangency point since $g_n(1,0) \neq 0$. So in this case, $m(\mathcal{F}, L) = n$.

(ii) There is a largest $0 \leq j < n$ such that $x \mapsto g_j(x,0)$ is not identically zero (otherwise $\sum_{k=0}^{n} g_k(x,0)$ would be everywhere zero, so $L \smallsetminus \operatorname{sing}(\mathcal{F})$ would be a leaf). This means that $g(x,0) = 0$ has exactly j roots counting multiplicities. In this case, the point at infinity for L is a tangency point of order $n - j$. In fact, (1.24) shows that $\tilde{g}(u,0) = -\sum_{k=0}^{n} u^{n-k} g_k(1,0) = -\sum_{k=0}^{j} u^{n-k} g_k(1,0)$, which has a root of multiplicity $n - j$ at $u = 0$. Thus, there are n tangency points on L altogether, so again $m(\mathcal{F}, L) = n$. $\square$

As an example, (1.12) shows that $\mathcal{F}$ *is induced by a holomorphic vector field on* $\mathbb{CP}^2$ *if and only if the geometric degree of* $\mathcal{F}$ *is* ≤ 1.

The set of all SHFC's on $\mathbb{CP}^2$ of geometric degree n is denoted by $\mathcal{D}_n$. Each $\mathcal{D}_n$ is topologized in the natural way: a neighborhood of $\mathcal{F} \in \mathcal{D}_n$ consists of all foliations of geometric degree n whose defining polynomials have coefficients close to that of $\mathcal{F}$, up to multiplication by a non-zero constant. To be more accurate, consider the complex linear space of all polynomial 1-forms ω as in (1.23). By Remark 1.14, ω and ω' define the same foliation if and only if there exists a non-zero constant λ such that $\omega' = \lambda\omega$. Therefore $\mathcal{D}_n$ can be considered as an open subset of the complex projective space $\mathbb{CP}^N$, where N is the dimension of the above linear space minus one, that is $N = 2\sum_{k=1}^{n+1} k + (n+1) - 1 = n^2 + 4n + 2$. It is not difficult to check that $\mathcal{D}_n$ is connected and dense in this projective space.

Corollary 1.19. *The set $\mathcal{D}_n$ of all SHFC's of geometric degree $n \geq 0$ on $\mathbb{CP}^2$ can be identified with an open, connected and dense subset of the complex projective space $\mathbb{CP}^N$, where $N = n^2 + 4n + 2$. So we can equip $\mathcal{D}_n$ with the induced topology and a natural Lebesgue measure class.*

The definition of $\mathcal{D}_n$ allows us to decompose the space $\mathcal{S}$ of all SHFC's on $\mathbb{CP}^2$ into a disjoint union $\bigcup \mathcal{D}_n$ and topologize it in a natural way. A subset $\mathcal{U}$ of $\mathcal{S}$ is open in this topology if and only if $\mathcal{U} \cap \mathcal{D}_n$ is open for every n. Hence this topology makes every $\mathcal{D}_n$ into a connected component of $\mathcal{S}$. Similarly, $\mathcal{S}$ inherits a natural Lebesgue measure class: A set $\mathcal{U} \subset \mathcal{S}$ has measure zero if and only if $\mathcal{U} \cap \mathcal{D}_n$ has measure zero in $\mathcal{D}_n$.

1.12. Line at Infinity as a Leaf. Let us find conditions on a 1-form ω which guarantee that the line at infinity $L_0 = \mathbb{CP}^2 \smallsetminus U_0$ with singular points of $\mathcal{F}_\omega$ deleted is a leaf. Consider an SHFC $\mathcal{F} \in \mathcal{D}_n$ induced by a polynomial 1-form ω as (1.23):

$\omega = (f + xh)dy - (g + yh)dx$, where $f = \sum_{k=0}^{n} f_k$, $g = \sum_{k=0}^{n} g_k$ and h is either a non-zero homogeneous polynomial of degree n, or $h \equiv 0$ but $yf_n - xg_n \not\equiv 0$.

THEOREM 1.20. *The line at infinity L_0 with singular points of $\mathcal{F} \in \mathcal{D}_n$ deleted is a leaf of $\mathcal{F}$ if and only if $h \equiv 0$.*

PROOF. This follows easily from the proof of Theorem 1.18. In fact, $L_0 \smallsetminus \text{sing}(\mathcal{F})$ is a leaf if and only if the line $\{u = 0\}$ is a solution of $\omega' = \tilde{f}dv - \tilde{g}du = 0$. By (1.24), this happens if and only if $h(1, v) \equiv 0$. Since h is a homogeneous polynomial, the latter condition is equivalent to $h \equiv 0$. □

REMARK 1.21. Here is an alternative notation for polynomial 1-forms which will be used in many subsequent discussions. Let

$$\omega = (f + xh)dy - (g + yh)dx = Pdy - Qdx,$$

as in (1.23). Define

$$R(x, y) = yP(x, y) - xQ(x, y) = yf(x, y) - xg(x, y)$$

as in §1.5, and note that deg $R \leq n + 1$. Then $\mathcal{F}|_{U_1}$ is given by $\{\omega' = 0\}$, where

$$\omega'(u, v) = u^k P\left(\frac{1}{u}, \frac{v}{u}\right) dv - u^k R\left(\frac{1}{u}, \frac{v}{u}\right) du,$$

and, as in (1.7), k is the least positive integer which makes ω' a polynomial 1-form. (Note that this representation should be identical to (1.24) up to a multiplicative constant.) If $h \not\equiv 0$, then deg $P = n + 1$ and so $k = n + 1$. On the other hand, if $h \equiv 0$, then $yf_n - xg_n \not\equiv 0$ by Theorem 1.18 which means deg $R = n + 1$. So again we have $k = n + 1$.

For a fixed $\mathcal{F}$, we denote $L_0 \smallsetminus \text{sing}(\mathcal{F})$ by $\mathcal{L}_\infty$ whenever it is a leaf of $\mathcal{F}$. We often refer to $\mathcal{L}_\infty$ as the *leaf at infinity.*

As can be seen from the above theorem, for a foliation $\mathcal{F} \in \mathcal{D}_n$ the line at infinity $L_0 \smallsetminus \text{sing}(\mathcal{F})$ is unlikely to be a leaf since this condition is equivalent to vanishing of a homogeneous polynomial. This is a consequence of the way we topologized the space $\mathcal{S}$ of all SHFC's on $\mathbb{CP}^2$ using the topologies on the $\mathcal{D}_n$. The decomposition $\bigcup \mathcal{D}_n$ is quite natural from the geometric point of view; however, it leads to a rather peculiar condition on the polynomials describing the associated 1-form (Theorem 1.18). The situation can be changed in a delicate way by choosing a different decomposition $\mathcal{S} = \bigcup \mathcal{A}_n$ which is more natural from the point of view of differential equations in $\mathbb{C}^2$ but has no longer an intrinsic geometric meaning. Elements of $\mathcal{A}_n$ are simply determined by the maximum degree of their defining polynomials.

DEFINITION 1.22. Fix the affine chart $(x, y) \in U_0$ and let $n \geq 0$. We say that an SHFC $\mathcal{F}$ belongs to the class $\mathcal{A}_n$ if it is induced by a polynomial 1-form $\omega = Pdy - Qdx$ with $\max\{\deg P, \deg Q\} = n$ and P, Q relatively prime. The number n is called the *affine degree* of $\mathcal{F}$ (with respect to U_0)

Note that $\mathcal{A}_n$ is well-defined since if $\mathcal{F}_\omega = \mathcal{F}_{\omega'}$, then $\omega' = \lambda\omega$ for some non-zero constant λ. It is important to realize that unlike the condition $\mathcal{F} \in \mathcal{D}_n$ (normal form (1.23)), whether or not $\mathcal{F} \in \mathcal{A}_n$ strongly depends on the choice of a particular affine coordinate system, and that is why we call n the "affine degree."

Consider the complex linear space of all polynomial 1-forms $\omega = Pdy - Qdx$ with $\max\{\deg P, \deg Q\} \leq n$, which has dimension $(n+1)(n+2)$. Then, as in the case of $\mathcal{D}_n$, one has

COROLLARY 1.23. *The set $\mathcal{A}_n$ of all SHFC's of affine degree $n \geq 0$ on $\mathbb{CP}^2$ can be identified with an open, connected and dense subset of the complex projective space $\mathbb{CP}^N$, where $N = n^2 + 3n + 1$. So we can equip $\mathcal{A}_n$ with the induced topology and a natural Lebesgue measure class.*

Using the decomposition of $\mathcal{S}$ into the disjoint union of the $\mathcal{A}_n$, we can define a new topology and measure class on $\mathcal{S}$ in the same way we did using the $\mathcal{D}_n$ (see the remarks after Corollary 1.19). In this new topology, each class $\mathcal{A}_n$ turns into a connected component of $\mathcal{S}$. The topologies and measure classes coming from $\bigcup \mathcal{A}_n$ and $\bigcup \mathcal{D}_n$ are significantly different. As a first indication of this difference, let $\mathcal{F} : \{\omega = Pdy - Qdx = 0\} \in \mathcal{A}_n$ and decompose $P = \sum_{k=0}^n P_k$ and $Q = \sum_{k=0}^n Q_k$ into the sum of the homogeneous polynomials P_k and Q_k of degree k. Then it easily follows from Theorem 1.18 and Theorem 1.20 that

COROLLARY 1.24. *The line at infinity $L_0 \smallsetminus \operatorname{sing}(\mathcal{F})$ is a leaf of $\mathcal{F} : \{Pdy - Qdx = 0\} \in \mathcal{A}_n$ if and only if $yP_n - xQ_n \not\equiv 0$.*

One concludes that in $\mathcal{A}_n$ it is very likely to have $L_0 \smallsetminus \operatorname{sing}(\mathcal{F})$ as a leaf, contrary to what we observed before in $\mathcal{D}_n$.

COROLLARY 1.25. *Fix the affine chart $(x, y) \in U_0$ and an SHFC $\mathcal{F} \in \mathcal{A}_n$.*

- *If the line at infinity $L_0 \smallsetminus \operatorname{sing}(\mathcal{F})$ is a leaf, then $\mathcal{F} \in \mathcal{A}_n \cap \mathcal{D}_n$ so that*
$$\text{affine degree of } \mathcal{F} = \text{geometric degree of } \mathcal{F}.$$
- *Otherwise, $\mathcal{F} \in \mathcal{A}_n \cap \mathcal{D}_{n-1}$ so that*
$$\text{affine degree of } \mathcal{F} = (\text{geometric degree of } \mathcal{F}) + 1.$$

The main reason for the contrast between $\mathcal{A}_n$ and $\mathcal{D}_n$ is the fact that $\dim \mathcal{A}_n < \dim \mathcal{D}_n < \dim \mathcal{A}_{n+1}$. In fact, it is not hard to see that $\mathcal{D}_n \subset \mathcal{A}_n \cup \mathcal{A}_{n+1}$ and $\mathcal{A}_n \subset \mathcal{D}_{n-1} \cup \mathcal{D}_n$. Fig. 3 is an attempt to illustrate the first property while Fig. 4 is a schematic diagram of the set-theoretic relations between these classes.

EXAMPLE 1.26. The two SHFC's
$$\mathcal{F}_1 : \{xdy - ydx = 0\} \qquad \mathcal{F}_2 : \{ydy - xdx = 0\}$$
both belong to $\mathcal{A}_1$ so they both have affine degree 1. However, $\mathcal{F}_1$ belongs to $\mathcal{D}_0$ hence has geometric degree 0, while $\mathcal{F}_2$ belongs to $\mathcal{D}_1$ and so it has geometric degree 1. Note that the line at infinity is *not* a leaf of $\mathcal{F}_1$ but it *is* a leaf of $\mathcal{F}_2$.

EXAMPLE 1.27. As another example, let us illustrate how the topologies coming from the two decompositions $\mathcal{S} = \bigcup \mathcal{A}_n = \bigcup \mathcal{D}_n$ are different. Consider the two SHFC's
$$\mathcal{F} : \{x^2dy - y^2dx = 0\},$$
$$\mathcal{F}_\varepsilon : \{(x^2 + \varepsilon xy^2)dy - (y^2 + \varepsilon y^3)dx = 0\}.$$
As $\varepsilon \to 0$, $\mathcal{F}_\varepsilon \to \mathcal{F}$ in the topology induced by $\bigcup \mathcal{D}_n$ but $\mathcal{F}_\varepsilon$ does not converge in the topology induced by $\bigcup \mathcal{A}_n$.

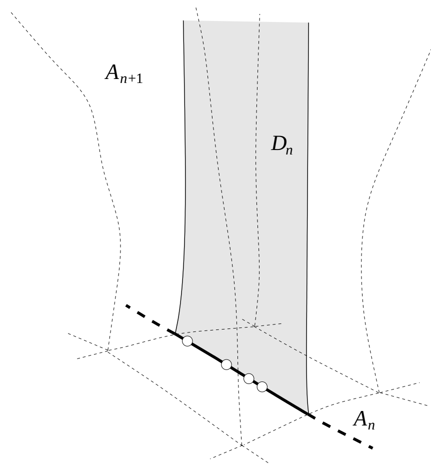

FIGURE 3. $\dim \mathcal{A}_n < \dim \mathcal{D}_n < \dim \mathcal{A}_{n+1}$.

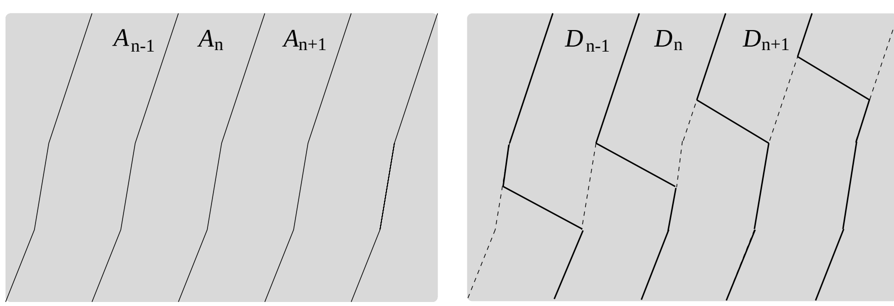

FIGURE 4. *Set-theoretic relations between* $\{\mathcal{D}_n\}$ *and* $\{\mathcal{A}_n\}$.

1.13. A Complex One-Dimensional Analogy. The following simple example may help understand the difference between the two decompositions $\{\mathcal{D}_n\}$ and $\{\mathcal{A}_n\}$: Let $\mathbf{S}$ be the space of all linear conjugacy classes of complex polynomial maps in $\mathbb{C}$ of degree at most 4 which are tangent to the identity map at the origin. This space can be naturally decomposed by the order of tangency near the fixed point at 0: For $0 \leq n \leq 2$, consider the sets $\mathbf{D}_n$ of conjugacy classes of normalized polynomials as follows:

$$\begin{aligned} \mathbf{D}_0 = & \ \langle z \mapsto z + z^4 \rangle \simeq \text{point} \\ \mathbf{D}_1 = & \ \langle z \mapsto z + z^3 + az^4 \rangle \simeq \mathbb{C} \\ \mathbf{D}_2 = & \ \langle z \mapsto z + z^2 + az^3 + bz^4 \rangle \simeq \mathbb{C}^2. \end{aligned}$$

Clearly $\mathbf{S} = \bigcup_{n=0}^{2} \mathbf{D}_n$ and this decomposition induces a topology $\tau_{\mathbf{D}}$ and a measure class $\mu_{\mathbf{D}}$ on $\mathbf{S}$. On the other hand, one can consider the following conjugacy classes determined by the degree of polynomials (i.e., by their behavior near infinity):

$$\begin{aligned} \mathbf{A}_0 = & \ \langle z \mapsto z + z^2 \rangle \simeq \text{point} \\ \mathbf{A}_1 = & \ \langle z \mapsto z + az^2 + z^3 \rangle \simeq \mathbb{C} \\ \mathbf{A}_2 = & \ \langle z \mapsto z + az^2 + bz^3 + z^4 \rangle \simeq \mathbb{C}^2. \end{aligned}$$

This gives rise to a second decomposition $\mathbf{S} = \bigcup_{n=0}^{2} \mathbf{A}_n$ hence a corresponding topology $\tau_{\mathbf{A}}$ and measure class $\mu_{\mathbf{A}}$ on $\mathbf{S}$. One has the relations

$$\begin{array}{ll} \mathbf{A}_0 \subset \mathbf{D}_2 & \mathbf{D}_0 \subset \mathbf{A}_2 \\ \mathbf{A}_1 \subset \mathbf{D}_1 \cup \mathbf{D}_2 & \mathbf{D}_1 \subset \mathbf{A}_1 \cup \mathbf{A}_2 \end{array}$$

The topologies $\tau_{\mathbf{D}}$ and $\tau_{\mathbf{A}}$ and measure classes $\mu_{\mathbf{D}}$ and $\mu_{\mathbf{A}}$ are very different. For example, $\mathbf{D}_1 \subset \mathbf{S}$ is an open set in $\tau_{\mathbf{D}}$, but it is *not* open in $\tau_{\mathbf{A}}$ since $\mathbf{D}_1 \cap \mathbf{A}_1$ is a single point. On the other hand, $\mathbf{D}_1 \subset \mathbf{A}_1 \cup \mathbf{A}_2$ has measure zero with respect to $\mu_{\mathbf{A}}$ but this is certainly not true with respect to $\mu_{\mathbf{D}}$.

1.14. Typical Properties. Certain geometric or dynamical properties often hold for "most" and not all SHFC's in $\mathcal{A}_n$ or $\mathcal{D}_n$. In these cases, we can use the Lebesgue measure class to make sense of this fact. A property $\mathcal{P}$ is said to be *typical* for elements of $\mathcal{A}_n$, or we say that a *typical* SHFC in $\mathcal{A}_n$ satisfies $\mathcal{P}$, if $\{\mathcal{F} \in \mathcal{A}_n : \mathcal{F} \text{ does not satisfy } \mathcal{P}\}$ has Lebesgue measure zero in $\mathcal{A}_n$. We can define a typical property in $\mathcal{D}_n$ in a similar way.

For example, it follows from Theorem 1.20 and Corollary 1.24 that *having the line at infinity as a leaf is not typical in $\mathcal{D}_n$ but it is typical in $\mathcal{A}_n$.*

Definition 1.28. Let $\mathcal{F} \in \mathcal{A}_n$. We say that $\mathcal{F}$ has *Petrovskiĭ-Landis property* if $L_0 \smallsetminus \text{sing}(\mathcal{F})$ is a leaf of $\mathcal{F}$ and $L_0 \cap \text{sing}(\mathcal{F})$ consists of $n+1$ distinct points. The class of all such $\mathcal{F}$ is denoted by $\mathcal{A}'_n$.

If $\mathcal{F} : \{Pdy - Qdx = 0\} \in \mathcal{A}'_n$, it follows from Corollary 1.24 that $yP_n - xQ_n \not\equiv 0$. On the other hand, using the notation of Remark 1.21, if $R = yP - xQ$, we have $\deg R = n+1$ and

$$L_0 \cap \text{sing}(\mathcal{F}) = \{(0,v) : u^{n+1} R\left(\frac{1}{u}, \frac{v}{u}\right)\Big|_{u=0} = 0\}$$

in the affine chart $(u,v) \in U_1$. It follows that $u^{n+1}R(1/u, v/u)|_{u=0}$ must have $n+1$ distinct roots in v. The above two conditions on P and Q show that

Corollary 1.29. *A typical SHFC in $\mathcal{A}_n$ has Petrovskiĭ-Landis property.*

2. The Monodromy Group of a Leaf

Given an SHFC $\mathcal{F}$ on $\mathbb{CP}^2$ one can study individual leaves as Riemann surfaces. However, to study the so-called *transverse dynamics* of the foliation one needs a tool to describe the rate of convergence or divergence of nearby leaves. The concept of holonomy, and in particular the monodromy of a leaf, first introduced by C. Ehresmann, is the essential tool in describing the transverse dynamics near the leaf. The point is that the transverse dynamics of a leaf depends directly on its fundamental group: the smaller $\pi_1(\mathcal{L})$ is, the simpler the behavior of the leaves near $\mathcal{L}$ will be.

2.1. Holonomy Mappings and Monodromy Groups. Let $\mathcal{F}$ be an SHFC on $\mathbb{CP}^2$ and $\mathcal{L}$ be a non-singular leaf of $\mathcal{F}$. Fix $p, q \in \mathcal{L}$, and consider small transversals $\Sigma, \Sigma' \simeq \mathbb{D}$ to $\mathcal{L}$ at p, q, respectively. Let $\gamma : [0,1] \to \mathcal{L}$ be a continuous path with $\gamma(0) = p, \gamma(1) = q$. For each $z \in \Sigma$ near p one can "travel" on $\mathcal{L}_z$ "over" $\gamma[0,1]$ to reach Σ' at some point z'. To be precise, let $\{(U_i, \varphi_i)\}_{0 \le i \le n}$ be foliation charts of $\mathcal{F}$ and $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ be a partition of $[0,1]$ such that if $U_i \cap U_j \neq \emptyset$ then $U_i \cup U_j$ is contained in a foliation chart, and $\gamma[t_i, t_{i+1}] \subset U_i$

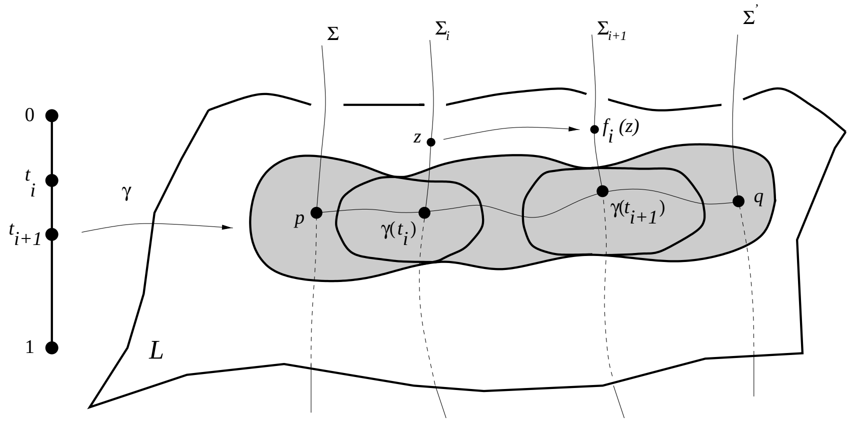

FIGURE 5. *Definition of holonomy.*

for $0 \leq i \leq n$. For each $1 \leq i \leq n$ choose a transversal $\Sigma_i \simeq \mathbb{D}$ to $\mathcal{L}$ at $\gamma(t_i)$, and set $\Sigma_0 = \Sigma$ and $\Sigma_{n+1} = \Sigma'$ (see Fig. 5). Then for each $z \in \Sigma_i$ sufficiently close to $\gamma(t_i)$ the plaque of U_i passing through z meets Σ_{i+1} in a unique point $f_i(z)$, and $z \mapsto f_i(z)$ is holomorphic, with $f_i(\gamma(t_i)) = \gamma(t_{i+1})$. It follows that the composition $f_\gamma := f_n \circ \cdots \circ f_0$ is defined for $z \in \Sigma$ near p, with $f_\gamma(p) = q$.

DEFINITION 2.1. The mapping f_γ is called the *holonomy* associated with γ.

There are several remarks about this mapping which can be checked directly from the definition (compare [**CL**]).

REMARK 2.2. f_γ is independent of the chosen transversals $\Sigma_i, 1 \leq i \leq n$, and the foliation charts U_i. Hence Σ, Σ', and γ determine the germ of f_γ at p.

REMARK 2.3. The germ of f_γ at p depends only on the homotopy class of γ rel$\{0, 1\}$, that is, if η is another path joining p and q in $\mathcal{L}$ which is homotopic to γ with $\eta(0) = \gamma(0)$ and $\eta(1) = \gamma(1)$, then the germ of f_η at p coincides with that of f_γ.

REMARK 2.4. If $\gamma^{-1}(t) := \gamma(1 - t)$, then $f_{\gamma^{-1}} = (f_\gamma)^{-1}$. In particular, f_γ represents the germ of a local *biholomorphism.*

REMARK 2.5. Let Σ_1 and Σ'_1 be other transversals to $\mathcal{L}$ at p and q, respectively. Let $h : \Sigma \to \Sigma_1$ and $\tilde{h} : \Sigma' \to \Sigma'_1$ be projections along the plaques of $\mathcal{F}$ in a neighborhood of p and q, respectively. Then the holonomy $g_\gamma : \Sigma_1 \to \Sigma'_1$ satisfies $g_\gamma = \tilde{h} \circ f_\gamma \circ h^{-1}$.

In the special case $p = q$, we obtain a generalization of the Poincaré first return map for real vector fields.

DEFINITION 2.6. Let $\mathcal{L}$ be a non-singular leaf of an SHFC $\mathcal{F}$ on $\mathbb{CP}^2$, let $p \in \mathcal{L}$, and let Σ be a transversal to $\mathcal{L}$ at p. For each $[\gamma] \in \pi_1(\mathcal{L}, p)$, the holonomy mapping $f_\gamma : \Sigma \to \Sigma$ is called the *monodromy mapping* of $\mathcal{L}$ associated with γ (Fig. 6).

Note that by Remark 2.3, the germ of f_γ at p depends only on the homotopy class $[\gamma]$. It is quite easy to see that $[\gamma] \mapsto f_\gamma$ is a homomorphism from $\pi_1(\mathcal{L}, p)$ into the group of germs of biholomorphisms of Σ fixing p: $[\gamma \circ \eta] \mapsto f_\gamma \circ f_\eta$.

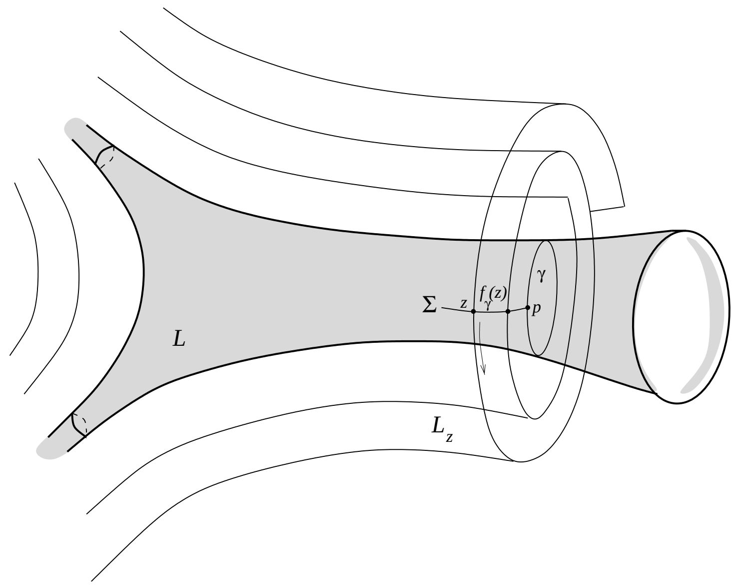

FIGURE 6. *Monodromy mapping associated with* γ.

REMARK 2.7. It should be noted that each $[\gamma] \in \pi_1(\mathcal{L}, p)$ determines only the *germ* at p of a biholomorphism f_γ of Σ, for changing γ in its homotopy class results in changing the domain of definition of f_γ.

REMARK 2.8. If the transversal Σ is replaced by another one Σ_1, then by Remark 2.5 there exists a local biholomorphism $h : \Sigma \to \Sigma_1$ fixing p such that the monodromy mapping $g_\gamma : \Sigma_1 \to \Sigma_1$ satisfies $g_\gamma = h \circ f_\gamma \circ h^{-1}$. In other words, the germ of the monodromy mapping f_γ depends only on $[\gamma]$ *up to conjugacy*. Since conjugate germs of biholomorphisms fixing p have the same iterative dynamics near p, it is not really important which transversal we choose at p.

CONVENTION 2.9. We always fix some $p \in \mathcal{L}$ as the base point for the fundamental group. We will fix some transversal Σ at p. Moreover, we will choose a coordinate on Σ in which $p = 0$. Thus, every monodromy mapping f_γ can be identified with an element of $\mathrm{Bih}_0(\mathbb{C})$, the group of germs at 0 of biholomorphisms $\mathbb{C} \to \mathbb{C}$ fixing the origin.

DEFINITION 2.10. The image of $\pi_1(\mathcal{L})$ under the monodromy mapping $[\gamma] \mapsto f_\gamma$ is called the *monodromy group* of $\mathcal{L}$, and is denoted by $G(\mathcal{L})$. Since we always fix the transversals, it can be identified with a subgroup of $\mathrm{Bih}_0(\mathbb{C})$.

Given a leaf $\mathcal{L}$ whose π_1 is finitely-generated, it is natural to fix some loops as the generators of $\pi_1(\mathcal{L})$. So we arrive at the following definition:

DEFINITION 2.11. A non-singular leaf $\mathcal{L}$ of an SHFC $\mathcal{F}$ on $\mathbb{CP}^2$ is called a *marked leaf* if $\pi_1(\mathcal{L})$ is finitely-generated and a set of loops $\{\gamma_1, \ldots, \gamma_k\}$ is given as its generators. Similarly, a finitely-generated subgroup $G \subset \mathrm{Bih}_0(\mathbb{C})$ is called a *marked subgroup* if a set of local biholomorphisms $\{f_1, \ldots, f_k\}$ is given as its generators.

Clearly, the monodromy group of a marked leaf is a marked subgroup if one chooses $\{f_{\gamma_1}, \cdots, f_{\gamma_k}\}$ as its generators.

2.2. The Monodromy Pseudo-Group of a Leaf. There is a certain difficulty in defining the orbit of a point under the action of the monodromy group $G(\mathcal{L})$ of a leaf $\mathcal{L}$. This is simply due to the fact that elements of $G(\mathcal{L})$ are *germs*, while in the definition of an orbit one should consider them as *maps*. On the other hand, given $f \in G(\mathcal{L})$, a point z may well be outside the domain of f but f may have an analytic continuation which is defined at z. Naturally, one askes to what extent this continuation really represents the same monodromy. For instance, if $\mathcal{F}$ is induced by a Hamiltonian form $\omega = dH$, then the monodromy group of the leaf at infinity $\mathcal{L}_\infty$ is abelian [**I3**]. Hence for every γ in the commutator subgroup $[\pi_1(\mathcal{L}_\infty), \pi_1(\mathcal{L}_\infty)]$ the germ of f_γ at 0 is the identity, which can be analytically continued everywhere. However, for z sufficiently far from 0, the monodromy f_γ may be undefined or it may differ from z.

Since the transverse dynamics of a leaf $\mathcal{L}$ is reflected in the orbit of points in the transversal Σ under the action of $G(\mathcal{L})$, it is quite natural to be careful about the domains of definitions. This point in addressed in the following definition:

Definition 2.12. Let $G \subset \mathrm{Bih}_0(\mathbb{C})$ be a marked subgroup with generators $\{f_1, \ldots, f_k\}$, all defined on some domain Ω around 0. The *pseudo-group PG* consists of all pairs (f, Ω_f), where $f \in G$ and Ω_f is a domain on which f is conformal, with the group operation $(f, \Omega_f) \circ (g, \Omega_g) := (f \circ g, \Omega_{f \circ g})$. The domain Ω_f is defined as follows: Let

$$f = \prod_{i=1}^{N} f_{j_i}^{\epsilon_i} \qquad , \; j_i \in \{1, \ldots, k\} \; , \; \epsilon_i \in \{-1, 1\} \tag{2.1}$$

be any representation of f in terms of the generators. Any germ $\prod_{i=1}^{n} f_{j_i}^{\epsilon_i}$, with $n \leq N$, is called an *intermediate representation* of f. The domain $\Omega_{\Pi f}$ associated to the representation (2.1) is defined as the maximal starlike domain centered at 0 contained in Ω on which all the intermediate representations can be analytically continued as conformal maps, with

$$\prod_{i=1}^{n} f_{j_i}^{\epsilon_i}(\Omega_{\Pi f}) \subset \Omega, \quad \text{for all} \quad n \leq N.$$

Finally, Ω_f is defined to be the union of $\Omega_{\Pi f}$'s for all possible representations of f of the form (2.1). It is clear that f is a conformal mapping on Ω_f.

The above definition associates a *monodromy pseudo-group $PG(\mathcal{L})$* to each marked leaf $\mathcal{L}$. The construction above allows us to define the *orbit* of $z \in \Sigma$ as $\{f(z) : (f, \Omega_f) \in PG(\mathcal{L}) \text{ and } z \in \Omega_f\}$. The basic property that we will be using is that the orbit of z under $PG(\mathcal{L})$ always lies in $\mathcal{L}_z \cap \Sigma$.

2.3. Multiplier of a Monodromy Mapping. As we will see later, the dynamics of an $f \in \mathrm{Bih}_0(\mathbb{C})$ is essentially dominated by its derivative $f'(0)$ at the fixed point 0, also known as the *multiplier* of f at 0. Therefore it would not be surprising that the behavior of leaves near a given leaf $\mathcal{L}$ is determined to a large extent by the multipliers at 0 of the monodromy mappings f_γ for $\gamma \in \pi_1(\mathcal{L})$. Our aim here is to give a formula for $f'_\gamma(0)$ in terms of a path integral (see [**KS**] for more general formulas of this type). We will use this formula in the next section, where

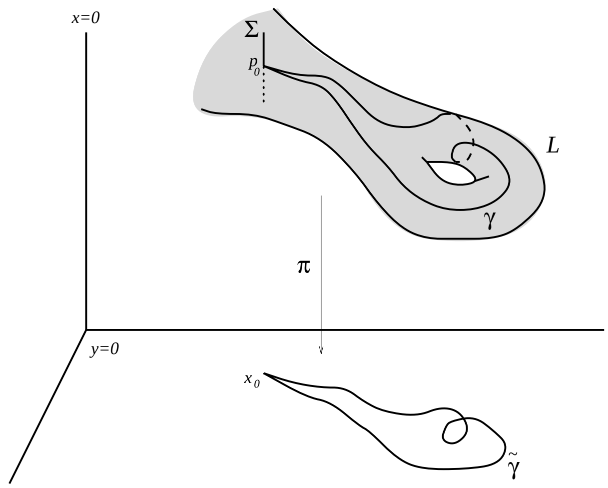

FIGURE 7. *Proof of Theorem 2.13.*

we will compute the multipliers of the monodromy mappings of the leaf at infinity of an $\mathcal{F} \in \mathcal{A}'_n$. It must be mentioned that this is quite similar to a well-known result of Reeb for real codimension 1 non-singular foliations [**Re**]. Of course a different argument is needed here since our foliations have real codimension 2. However, our proof is greatly facilitated by the fact the foliations are induced by polynomial 1-forms.

Let $\mathcal{F} : \{\omega = Pdy - Qdx = 0\} \in \mathcal{A}_n$, and suppose that $\mathcal{L}_\infty$, the line at infinity with sing($\mathcal{F}$) deleted, is a leaf of $\mathcal{F}$ (see Corollary 1.24). Fix some non-singular leaf $\mathcal{L} \neq \mathcal{L}_\infty$. It follows that $\mathcal{L}$ is completely contained in the affine chart $(x, y) \in U_0$. Without loss of generality we may assume that $E := \mathcal{L} \cap S_P$ is discrete in the leaf topology, hence countable (see §1.4 for the definition of S_P). In fact, if E is not discrete, it has a limit point z_0 in $\mathcal{L}$. Parametrizing $\mathcal{L}$ around z_0 by $T \mapsto (x(T), y(T))$ with $z_0 = (x(0), y(0))$, we conclude that $P(x(T), y(T)) \equiv 0$. By analytic continuation, $P(x, y) = 0$ for every $(x, y) \in \mathcal{L}$. Therefore we can pursue the argument with $\mathcal{L} \cap S_Q$, which is finite since P and Q are assumed to be relatively prime.

So assume that E is discrete. By integrability of ω, there exists a meromorphic 1-form $\alpha = Xdy - Ydx$, holomorphic on $\mathcal{L} \smallsetminus E$, such that $d\omega = \omega \wedge \alpha$ (since P and Q have no common factor, one can actually find *polynomial* vector fields X and Y with this property; see equation (2.3) below).

THEOREM 2.13. *Given* $\gamma \in \pi_1(\mathcal{L}, p_0)$ *one has*

$$f'_\gamma(0) = \exp\left(-\int_\gamma \alpha\right). \tag{2.2}$$

PROOF. Consider $\mathcal{L}$ as the graph of a (multi-valued) function $y = \varphi(x)$ over some region of the x-axis. Let $E = \mathcal{L} \cap S_P$ and $\tilde{E} := \{x \in \mathbb{C} : \text{There exists } y \in \mathbb{C} \text{ such that } (x, y) \in E\}$. Without loss of generality we can assume that the base point $p_0 = (x_0, y_0)$ is not in E. Moreover, we may replace γ by a path in its homotopy class that avoids E, if necessary. Take a "vertical" Σ transversal to $\mathcal{L}$ at p_0, and let y be the coordinate on Σ (Fig. 7). For $y \in \Sigma$ near y_0, each leaf $\mathcal{L}_y$ is the graph of the solution $\Phi(x,y)$ of $dy/dx = Q(x,y)/P(x,y)$, i.e.,

$$\frac{\partial \Phi}{\partial x}(x,y) = \frac{Q(x,\Phi(x,y))}{P(x,\Phi(x,y))}, \qquad \Phi(x_0,y) = y.$$

Define $\xi(x) := \partial\Phi/\partial y(x,y_0)$. Note that

$$\begin{aligned}
\frac{d\xi}{dx}(x) &= \frac{\partial^2 \Phi}{\partial x \partial y}(x,y_0) \\
&= \frac{\partial}{\partial y}\left[\frac{Q(x,\Phi(x,y))}{P(x,\Phi(x,y))}\right](x,y_0) \\
&= \left[\frac{Q_y(x,\varphi(x))P(x,\varphi(x)) - P_y(x,\varphi(x))Q(x,\varphi(x))}{P^2(x,\varphi(x))}\right]\xi(x) \\
&=: T(x)\xi(x),
\end{aligned}$$

with $\xi(x_0) = 1$. Thus $\xi(x) = \exp(\int_{x_0}^{x} T(\tau)d\tau)$, where the path of integration avoids $\tilde{E}$. But $f'_\gamma(0)$ is the result of analytic continuation of ξ along $\tilde{\gamma}$, the projection on the x-axis of γ, so that $f'_\gamma(0) = \exp(\int_{\tilde{\gamma}} T(x)dx)$. On the other hand, the condition $d\omega = \omega \wedge \alpha$ shows that

$$YP - XQ = P_x + Q_y, \tag{2.3}$$

so that on an open neighborhood $W \subset \mathcal{L}$ of γ one has

$$\alpha|_W = (Xdy - Ydx)|_W = \left(\frac{XQ}{P} - Y\right)\Big|_W \, dx = -\left(\frac{P_x + Q_y}{P}\right)\Big|_{y=\varphi(x)} \, dx.$$

Now, compute

$$\begin{aligned}
\int_{\tilde{\gamma}} T(x)\,dx + \int_\gamma \alpha &= \int_{\tilde{\gamma}} \left[T(x) - \left(\frac{P_x + Q_y}{P}\right)\Big|_{y=\varphi(x)}\right] dx \\
&= \int_{\tilde{\gamma}} \frac{Q_y P - P_y Q - P(P_x + Q_y)}{P^2}\Big|_{y=\varphi(x)} \, dx \\
&= -\int_{\tilde{\gamma}} \frac{P_y Q + P_x P}{P^2}\Big|_{y=\varphi(x)} \, dx \\
&= -\int_{\tilde{\gamma}} \frac{\partial/\partial x(P(x,\varphi(x)))}{P} \, dx \\
&= -\int_{\tilde{\gamma}} \frac{\partial}{\partial x} \log P(x,\varphi(x)) \, dx \\
&= 2\pi i n
\end{aligned}$$

for some integer n by the Argument Principle. Hence $\int_{\tilde{\gamma}} T(x)\,dx$ and $-\int_\gamma \alpha$ differ by an integer multiple of $2\pi i$, which proves the result. $\square$

2.4. Monodromy Group of the Leaf at Infinity. Let $\mathcal{F} \in \mathcal{A}'_n$ be an SHFC on $\mathbb{CP}^2$ having Petrovskiĭ-Landis property. Recall from Definition 1.28 that this means $L_0 \smallsetminus \operatorname{sing}(\mathcal{F})$ is a leaf and $L_0 \cap \operatorname{sing}(\mathcal{F})$ consists of exactly $n+1$ points $\{p_1, \ldots, p_{n+1}\}$. Having an algebraic leaf homeomorphic to the punctured Riemann sphere imposes severe restrictions on the global behavior of the leaves. Roughly speaking, for $\mathcal{F} \in \mathcal{A}'_n$ the global behavior of the leaves is essentially determined by their local behavior in some neighborhood of the leaf at infinity $\mathcal{L}_\infty$. In fact, under the assumption $\mathcal{F} \in \mathcal{A}'_n$, each non-singular leaf must accumulate on L_0 (Corollary 2.17 below). Once this is established, we can study the monodromy group of $\mathcal{L}_\infty$ to get information about the behavior of nearby leaves, which then can be transferred elsewhere.

Theorem 2.14. *Let $X = \sum_{j=1}^n f_j \ \partial/\partial z_j$ be a holomorphic vector field on $\mathbb{C}^n$. Then every non-singular solution of the differential equation $dz/dT = X(z)$ is unbounded.*

Proof. Fix $z_0 \in \mathbb{C}^n$ with $X(z_0) \neq 0$, and suppose by way of contradiction that the integral curve $\mathcal{L}_{z_0}$ passing through z_0 is bounded. Let $T \mapsto \eta(T)$ be a local parametrization of $\mathcal{L}_{z_0}$, with $\eta(0) = z_0$. Let $R > 0$ be the largest radius such that $\eta(T)$ can be analytically continued over $\mathbb{D}(0, R)$. If $R = +\infty$, then $\eta(T)$ will be constant by Liouville's Theorem, contrary to the assumption $X(z_0) \neq 0$. So $R < +\infty$. Recall from the Existence and Uniqueness Theorem of solutions of holomorphic differential equations that for each $T_0 \in \mathbb{C}$ and each $p \in \mathbb{C}^n$, if X is holomorphic on $\{z \in \mathbb{C}^n : |z - p| < b\}$, then there exists a local parametrization $T \mapsto \eta_p(T)$ of $\mathcal{L}_p$, with $\eta_p(T_0) = p$, defined on $\mathbb{D}(T_0, b/(M + kb))$, where $M = \sup\{|X(z)| : |z - p| < b\}$ and $k = \sup\{|dX(z)/dz| : |z - p| < b\}$ (see for example [**CoL**]). In our case, since $\mathcal{L}_{z_0}$ is bounded by the assumption, both $|X(\cdot)|$ and $|dX(\cdot)/dz|$ will be bounded on $\mathcal{L}_{z_0}$, and hence the quantity $b/(M + kb)$ has a uniform lower bound $2\delta > 0$ for all $p \in \mathcal{L}_{z_0}$. It follows that for each $T_0 \in \mathbb{D}(0, R)$ with $R - |T_0| < \delta$, we can find a local parametrization $\eta_p : \mathbb{D}(T_0, 2\delta) \to \mathbb{C}^n$ for $\mathcal{L}_{z_0}$ around $p = \eta_p(T_0)$. These local parametrizations patch together, giving an analytic continuation of η over the disk $\mathbb{D}(0, R + \delta)$. The contradiction shows that $\mathcal{L}_{z_0}$ must be unbounded. $\square$

Remark 2.15. The same argument in the real case gives another proof of the fact that an integral curve of a differential equation on $\mathbb{R}^n$ is either unbounded or it is bounded and parametrized by the entire real line.

Corollary 2.16. *Let $L \subset \mathbb{CP}^2$ be any projective line and $\mathcal{L}$ be any non-singular leaf of an SHFC $\mathcal{F}$. Then $\overline{\mathcal{L}} \cap L \neq \emptyset$.*

Proof. Consider the affine chart (x, y) for $\mathbb{CP}^2 \smallsetminus L \simeq \mathbb{C}^2$. In this coordinate system, $L = L_0$. Note that the leaves of $\mathcal{F}|_{\mathbb{C}^2}$ are the integral curves of a (polynomial) vector field. Now the result follows from Theorem 2.14. $\square$

Corollary 2.17. *Any non-singular leaf of an SHFC in $\mathcal{A}'_n$ has an accumulation point on the line at infinity.*

If the accumulation point is not singular, then the whole line at infinity is contained in the closure of the leaf by Proposition 1.6.

Knowing that each leaf accumulates on L_0, we now proceed to study the monodromy group of the leaf at infinity $\mathcal{L}_\infty$. Let $\mathcal{F} : \{\omega = Pdy - Qdx = 0\} \in \mathcal{A}'_n$ and

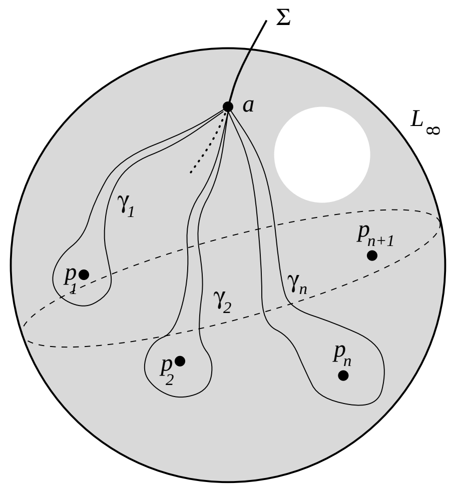

FIGURE 8. *Marking the leaf at infinity.*

$L_0 \cap \operatorname{sing}(\mathcal{F}) = \{p_1, \ldots, p_{n+1}\}$. Recall from Remark 1.21 that $\mathcal{F}$ is described in the affine chart $(u, v) \in U_1$ by

$$\omega'(u,v) = u^{n+1} P\left(\frac{1}{u}, \frac{v}{u}\right) dv - u^{n+1} R\left(\frac{1}{u}, \frac{v}{u}\right) du = 0, \tag{2.4}$$

where $R(x, y) = yP(x, y) - xQ(x, y)$. The line at infinity L_0 is the closure of $\{(0, v) : v \in \mathbb{C}\}$, and $p_j := (0, a_j)$, $1 \leq j \leq n+1$, where the a_j are distinct roots in v of the polynomial $u^{n+1}R(1/u, v/u)|_{u=0}$.

The leaf at infinity $\mathcal{L}_\infty = L_0 \smallsetminus \{p_1, \ldots, p_{n+1}\}$ can be made into a marked leaf by choosing fixed loops $\{\gamma_1, \ldots, \gamma_n\}$ as generators of $\pi_1(\mathcal{L}_\infty)$. Fixing a base point $a \notin \{p_1, \ldots, p_{n+1}\}$, each γ_j goes around p_j once in the positive direction and does not encircle p_j for $i \neq j$ (Fig. 8).

DEFINITION 2.18. The monodromy mappings f_{γ_j} for $1 \leq j \leq n$ generate the *monodromy group of the leaf at infinity*, denoted by G_∞.

Let us compute the multiplier of each monodromy mapping f_{γ_j} in terms of P and Q. In the affine chart $(u, v) \in U_1$, the foliation is induced by the polynomial 1-form ω' of (2.4). Write

$$u^{n+1} P\left(\frac{1}{u}, \frac{v}{u}\right) =: u\tilde{P}(u,v) \qquad \text{and} \qquad u^{n+1} R\left(\frac{1}{u}, \frac{v}{u}\right) =: \tilde{R}(u,v), \tag{2.5}$$

where $\tilde{P}$ and $\tilde{R}$ are polynomials with $\tilde{R}(0, a_j) = 0$ for $1 \leq j \leq n+1$. Note that $\mathcal{F}|_{U_1}$ is induced by the vector field

$$X_1 = u\tilde{P}(u,v)\frac{\partial}{\partial u} + \tilde{R}(u,v)\frac{\partial}{\partial v}$$

(compare (1.7)). Let us consider the Jacobian matrix DX_1 at the singular point p_j:

$$DX_1(p_j) = \begin{pmatrix} \tilde{P}(0, a_j) & 0 \\ \tilde{R}_u(0, a_j) & \tilde{R}_v(0, a_j) \end{pmatrix}, \tag{2.6}$$

where the indices denote partial derivatives. The quotient

$$\lambda_j := \frac{\tilde{P}(0,a_j)}{\tilde{R}_v(0,a_j)} \tag{2.7}$$

of the eigenvalues of the matrix (2.6) is called the *characteristic number* of the singularity p_j. Note that since the roots of $\tilde{R}(0,v)$ are simple by assumption, we have $\tilde{R}_v(0,a_j) \neq 0$ and λ_j of (2.7) is well-defined. On the other hand, the characteristic number is evidently independent of the vector field representing $\mathcal{F}$ near p_j, since such a vector field is unique up to multiplication by a nowhere vanishing holomorphic function near p_j.

For simplicity we denote f_{γ_j} by f_j and $f_j'(0)$ by ν_j.

PROPOSITION 2.19. $\nu_j = e^{2\pi i\lambda_j}$.

PROOF. By Theorem 2.13, we have $\nu_j = \exp(-\int_{\gamma_j}\alpha)$, where α is any meromorphic 1-form which satisfies $d\omega' = \omega' \wedge \alpha$. Choose, for example, the 1-form $\alpha = -(u\tilde{P}_u + \tilde{P} + \tilde{R}_v)/\tilde{R}\ dv$. Then we have

$$\begin{aligned} -\int_{\gamma_j}\alpha &= \int_{\gamma_j} \left.\frac{u\tilde{P}_u + \tilde{P} + \tilde{R}_v}{\tilde{R}}\right|_{u=0} dv \\ &= \int_{\gamma_j} \frac{\tilde{P}(0,v) + \tilde{R}_v(0,v)}{\tilde{R}(0,v)}\ dv \\ &= 2\pi i \ \mathrm{Res}\left[\frac{\tilde{P}(0,v) + \tilde{R}_v(0,v)}{\tilde{R}(0,v)}; a_j\right] \\ &= 2\pi i(\lambda_j + 1), \end{aligned}$$

so that $\exp(-\int_{\gamma_j}\alpha) = e^{2\pi i\lambda_j}$. $\square$

REMARK 2.20. Since $f_1\circ\cdots\circ f_{n+1} =$id, one has $\nu_1\cdots\nu_{n+1} = 1$ so that $\sum_{j=1}^{n+1}\lambda_j$ is an integer by the above proposition. This integer turns out to be 1 by the following argument. By (2.7), λ_j is the residue at a_j of the meromorphic function $\tilde{P}(0,v)/\tilde{R}(0,v)$ on $L_0 \simeq \mathbb{CP}^1$. If $\tilde{R}(0,v) = c\prod_{j=1}^{n+1}(v-a_j)$, then c is the coefficient of y^{n+1} in $R(x,y)$ by (2.5), hence it is the coefficient of y^n in $P(x,y)$ since $R = yP - xQ$. So again by (2.5) $\tilde{P}(0,v)$ is a polynomial in v with leading term cv^n. It follows that the residue at infinity of $\tilde{P}(0,v)/\tilde{R}(0,v)$ is -1. Hence $\sum_{j=1}^{n+1}\lambda_j - 1 = 0$ by the Residue Theorem.

2.5. Equivalence of Foliations and Subgroups of $\mathbf{Bih}_0(\mathbb{C})$. For singular smooth 1-dimensional foliations on real manifolds, one can speak of topological or C^k *equivalences*, or topological or C^k *conjugacies*. In the case of an equivalence, one is concerned only about the topology of the leaves, but in the case of a conjugacy, the actual parametrization of the leaves is also relevant. Of course the latter makes sense only when the foliations are described by single smooth vector fields on the ambient space.

In the complex analytic case, we can still think of equivalences as well as conjugacies between SHFC's. But again the notion of conjugacy requires the existence of holomorphic vector fields defined on the ambient space representing the foliations.

One can develop a theory of conjugacies for holomorphic vector fields defined on non-compact complex manifolds such as open subsets of $\mathbb{C}^n$. However, the study of conjugacies for holomorphic vector fields on compact complex manifolds is certainly less interesting, as the possible examples are rather rare and often trivial (compare Proposition 1.9).

Based on this observation, the most natural notion of equivalence between SHFC's on $\mathbb{CP}^2$ seems to be the following:

DEFINITION 2.21. Two SHFC's $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{CP}^2$ are said to be *topologically* (resp. *holomorphically*) *equivalent* if there exists a homeomorphism (resp. biholomorphism) $H : \mathbb{CP}^2 \to \mathbb{CP}^2$ which maps the leaves of $\mathcal{F}$ to those of $\mathcal{F}'$.

The existence of an equivalence between two SHFC's has the following implication on the monodromy groups:

PROPOSITION 2.22. *Let $\mathcal{F}$ and $\mathcal{F}'$ be topologically (resp. holomorphically) equivalent SHFC's linked by $H : \mathbb{CP}^2 \to \mathbb{CP}^2$. Let p be a non-singular point for $\mathcal{F}$. Then the monodromy group $G(\mathcal{L}_p)$ is isomorphic to $G(\mathcal{L}'_{H(p)})$. More precisely, there exists a local homeomorphism (resp. biholomorphism) h defined on some neighborhood of $0 \in \mathbb{C}$, with $h(0) = 0$, and a group isomorphism $k : G(\mathcal{L}_p) \to G(\mathcal{L}'{}_{H(p)})$ such that $h \circ f = k(f) \circ h$ for every $f \in G(\mathcal{L}_p)$.*

PROOF. Let $\gamma \in \pi_1(\mathcal{L}_p, p)$ and Σ be a transversal to $\mathcal{L}_p$ at p. Set $q = H(p)$, $\gamma' := H_*\gamma \in \pi_1(\mathcal{L}'_q, q)$, and let Σ' be a transversal to $\mathcal{L}'_q$ at q. As in the definition of the monodromy mapping, choose foliation charts $\{(U_i, \varphi_i)\}_{0 \leq i \leq n}$ and transversals Σ_i for $\mathcal{F}$, and the corresponding data $\{(U'_i, \varphi'_i)\}_{0 \leq i \leq n}$ and Σ'_i for $\mathcal{F}'$, and consider the decompositions $f_\gamma = f_n \circ \cdots \circ f_0$ and $g_{\gamma'} = g_n \circ \cdots \circ g_0$ (compare §2.1). Without loss of generality we may assume that $U'_i = H(U_i)$. Since $\mathcal{F}|_{U_i}$ (resp. $\mathcal{F}'|_{U'_i}$) is a trivial foliation, one has a projection along leaves $\pi_i : U_i \to \Sigma_i$ (resp. $\pi'_i : U'_i \to \Sigma'_i$) which sends every z to the unique intersection point of the plaque of U_i (resp. U'_i) through z with Σ_i (resp. Σ'_i).

Define $h_i : \Sigma_i \to \Sigma'_i$ by $h_i := \pi'_i \circ H$. Since H is a leaf-preserving homeomorphism (resp. biholomorphism) each h_i is also a homeomorphism (resp. biholomorphism) with inverse $h_i^{-1} = \pi_i \circ H^{-1}$ (see Fig. 9). Now the definition of f_i and g_i shows that $h_{i+1} \circ f_i = g_i \circ h_i$ for $0 \leq i \leq n$. Therefore, the two relations $f_\gamma = f_n \circ \cdots \circ f_0$ and $g_{\gamma'} = g_n \circ \cdots \circ g_0$ show that $g_{\gamma'} \circ h_0 = h_0 \circ f_\gamma$. To complete the proof, note that the mapping $f_\gamma \mapsto k(f_\gamma) := h_0 \circ f_\gamma \circ {h_0}^{-1}$ is an isomorphism between $G(\mathcal{L}_p)$ and $G(\mathcal{L}'_q)$. □

The following definition is suggested by the above proposition:

DEFINITION 2.23. Two subgroups $G, G' \subset \mathrm{Bih}_0(\mathbb{C})$ are *topologically* (resp. *holomorphically*) *equivalent* if there exists a homeomorphism (resp. biholomorphism) h defined on some neighborhood of $0 \in \mathbb{C}$, with $h(0) = 0$, such that $h \circ f \circ h^{-1} \in G'$ if and only if $f \in G$.

It follows that the mapping $f \mapsto k(f) := h \circ f \circ h^{-1} : G \to G'$ is an isomorphism, and the following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{C}, 0) & \xrightarrow{f} & (\mathbb{C}, 0) \\ \big\downarrow h & & \big\downarrow h \\ (\mathbb{C}, 0) & \xrightarrow{k(f)} & (\mathbb{C}, 0) \end{array} \tag{2.8}$$

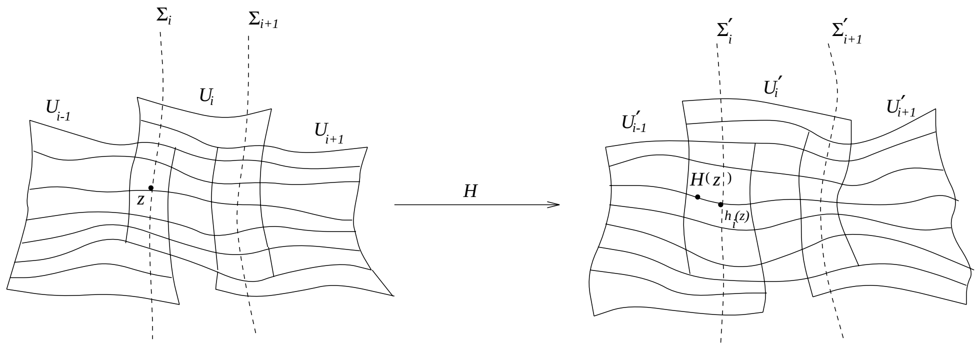

FIGURE 9. *Equivalent foliations.*

THEOREM 2.24. *If two SHFC's $\mathcal{F}$ and $\mathcal{F}'$ on $\mathbb{CP}^2$ are topologically (resp. holomorphically) equivalent, then so are the monodromy groups of the corresponding leaves.*

This theorem naturally leads us to the study of equivalent subgroups of $\mathrm{Bih}_0(\mathbb{C})$ in order to gain information about equivalent SHFC's. Below we list a few important results concerning these equivalence problems.

THEOREM 2.25. *Let $G, G' \subset \mathrm{Bih}_0(\mathbb{C})$ be two topologically equivalent marked subgroups linked by an orientation-preserving homeomorphism h such that the diagram (2.8) is commutative. Suppose that G is non-abelian, and that there exist $f_1, f_2 \in G$ such that the multiplicative subgroup of $\mathbb{C}^*$ generated by $f_1'(0), f_2'(0)$ is dense in $\mathbb{C}$. Then h is actually a biholomorphism, so that G and G' are holomorphically equivalent.*

This phenomenon is called *absolute rigidity* of subgroups of $\mathrm{Bih}_0(\mathbb{C})$. (A subgroup $G \subset \mathrm{Bih}_0(\mathbb{C})$ is called *absolutely rigid* if every subgroup which is topologically equivalent to G is holomorphically equivalent to it.) The proof of this theorem can be found in [**I3**].

Along this line, A. Shcherbakov [**Sh**] has shown the following result:

THEOREM 2.26. *A non-solvable subgroup of* $\mathrm{Bih}_0(\mathbb{C})$ *is absolutely rigid.*

For an almost complete topological and analytical classification of germs in $\mathrm{Bih}_0(\mathbb{C})$, see paper II of [**I5**].

The next result gives "moduli of stability" for typical SHFC's on $\mathbb{CP}^2$.

THEOREM 2.27. *Let $\mathcal{F}, \mathcal{F}' \in \mathcal{A}_n'$ be two SHFC's on $\mathbb{CP}^2$ having no algebraic leaves other than the leaf at infinity. Let $L_0 \cap \mathrm{sing}(\mathcal{F}) = \{p_1, \ldots, p_{n+1}\}$, $L_0 \cap \mathrm{sing}(\mathcal{F}') = \{p_1', \ldots, p_{n+1}'\}$, and λ_j and λ_j' be the characteristic numbers of p_j and p_j', respectively. Suppose that λ_j and λ_j' are non-zero, and $\mathcal{F}$ and $\mathcal{F}'$ are topologically equivalent by a homeomorphism $H : \mathbb{CP}^2 \to \mathbb{CP}^2$ with $H(p_j) = p_j'$. Then there exists an $\mathbb{R}$-linear transformation $A : \mathbb{C} \to \mathbb{C}$, with $A(\lambda_j) = \lambda_j'$ for $1 \leq j \leq n+1$.*

It follows in particular that the $n+1$ characteristic numbers λ_j' span a subspace of $\mathbb{C}^{n+1}$ of dimension ≤ 2. The proof of Theorem 2.27 is based on studying the equivalence of the monodromy groups of the leaves at infinity, and uses the same techniques as the proof of Theorem 2.25. It was first proved by Il'yashenko [**I3**] in the case λ_j, λ'_j are not real numbers. Later, it was generalized by V. Naishul [**N**], who presented a much more difficult argument to handle the case where the characteristic numbers are real.

Recall that $\mathcal{F} \in \mathcal{A}_n$ is *structurally stable* if there exists a neighborhood $\Omega \subset \mathcal{A}_n$ of $\mathcal{F}$ such that every SHFC in Ω is topologically equivalent to $\mathcal{F}$. Since it can be shown that the set of $\mathcal{F} \in \mathcal{A}'_n$ which do not have any algebraic leaf other than $\mathcal{L}_\infty$ is open and dense in $\mathcal{A}_n$ (see Definition 1.28 and the proof of Proposition 3.21), it follows from Theorem 2.27 that

Corollary 2.28. *No SHFC in $\mathcal{A}_n$ is structurally stable when $n \geq 2$.*

The following theorem, which proves a type of "absolute rigidity" for SHFC's, is a fundamental result first proved by Il'yashenko [**I3**].

Theorem 2.29. *A typical $\mathcal{F} \in \mathcal{A}_n$ is absolutely rigid. That is, there exist neighborhoods $\Omega \subset \mathcal{A}_n$ of $\mathcal{F}$ and U of the identity mapping on $\mathbb{CP}^2$ in the uniform topology such that every SHFC in Ω which is topologically equivalent to $\mathcal{F}$ by a homeomorphism in U is holomorphically equivalent to $\mathcal{F}$.*

X. Gómez-Mont [**GO**] has generalized the above theorem to SHFC's on projective complex surfaces which have an algebraic leaf of sufficiently rich homotopy group.

3. Density and Ergodicity Theorems

In the previous chapter we noted that the behavior of leaves near the leaf at infinity $\mathcal{L}_\infty$ gives us information about their global behavior. The orbits of points under the action of the monodromy pseudo-group PG_∞ in turn give us a picture of the behavior of leaves near $\mathcal{L}_\infty$. So a natural task is to consider dynamics of germs in G_∞, that is, the iterations in a finitely-generated subgroup of $\mathrm{Bih}_0(\mathbb{C})$.

Here is a sketch of what will follow in this chapter. First we consider elements of $\mathrm{Bih}_0(\mathbb{C})$ without any attention to the relationship with the monodromy groups of SHFC's. We study the linearization of hyperbolic germs (Theorem 3.2), and approximation of a linear map by elements of a pseudo-group of germs in $\mathrm{Bih}_0(\mathbb{C})$ (Proposition 3.4) which leads us to a local density theorem (Theorem 3.5). We then consider the notion of ergodicity in $\mathrm{Bih}_0(\mathbb{C})$ and find conditions under which a finitely-generated subgroup of $\mathrm{Bih}_0(\mathbb{C})$ is ergodic (Theorem 3.15). Finally, these results will be applied to the monodromy group G_∞ of a typical $\mathcal{F} \in \mathcal{A}_n$, leading to the density theorem of M. Khudai-Veronov (Theorem 3.24) and the ergodicity theorem of Il'yashenko and Sinai (Theorem 3.26).

3.1. Linearization of Germs in $\mathrm{Bih}_0(\mathbb{C})$.

Let us begin with the following definition:

Definition 3.1. A germ $f \in \mathrm{Bih}_0(\mathbb{C})$ is called *linearizable* if there exists a holomorphic change of coordinate $\zeta = \zeta(z)$ near 0, with $\zeta(0) = 0$, such that

$$\zeta(f(z)) = f'(0) \cdot \zeta(z). \tag{3.1}$$

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{C},0) & \xrightarrow{f} & (\mathbb{C},0) \\ \Big\downarrow \zeta & & \Big\downarrow \zeta \\ (\mathbb{C},0) & \xrightarrow{f'(0)\cdot} & (\mathbb{C},0) \end{array}$$

It follows that the holomorphic change of coordinate ζ conjugates f with its tangent map at the fixed point 0.

If the germs f and g are conjugate, say if $\zeta \circ f \circ \zeta^{-1} = g$, then $\zeta \circ f^n \circ \zeta^{-1} = g^n$ for every $n \geq 1$, so f and g have identical iterative dynamics near 0 up to a change of coordinate. Thus the possibility of linearization can be very helpful in understanding the dynamics of iterations.

It is a remarkable fact that the possibility of linearizing a germ depends on the multiplier of the germ at the fixed point 0. In particular, it was shown by G. Koenigs that $f \in \mathrm{Bih}_0(\mathbb{C})$ is linearizable if f is *hyperbolic* in the sense that $|f'(0)| \neq 1$ [**M**].

THEOREM 3.2. *Every hyperbolic germ $f \in \mathrm{Bih}_0(\mathbb{C})$ is linearizable. The local linearization ζ is unique up to multiplication by a non-zero constant.*

PROOF. Let $\nu := f'(0)$. Without losing generality we may assume that $|\nu| < 1$, for otherwise we can consider f^{-1}. Choose a constant $c < 1$ so that $c^2 < |\nu| < c$, and let $r > 0$ be such that $|f(z)| \leq c|z|$ for $z \in \mathbb{D}(0, r)$. For every $z_0 \in \mathbb{D}(0, r)$, the orbit $\{z_n := f^n(z_0)\}_{n \geq 0}$ converges geometrically towards the origin, with $|z_n| \leq rc^n$. Choose $k > 0$ so that $|f(z) - \nu z| \leq k|z|^2$ for all $z \in \mathbb{D}(0, r)$. It follows that $|z_{n+1} - \nu z_n| \leq kr^2 c^{2n}$. Set $\zeta_n(z) := f^n(z)/\nu^n$. Then $|\zeta_{n+1} - \zeta_n| \leq (kr^2/|\nu|)(c^2/|\nu|)^n$. This estimate shows that the sequence $\{\zeta_n\}$ converges uniformly on $\mathbb{D}(0, r)$ to a holomorphic map ζ. The identity $\zeta \circ f = \nu\zeta$ is immediate. Note that $\zeta'(0) = \lim \zeta_n'(0) = \lim \nu^{-n}(f^n)'(0) = 1$, so ζ is a local biholomorphism. If $\eta \circ f = \nu\eta$, then $\zeta \circ \eta^{-1}$ commutes with the linear map $z \mapsto \nu z$. Applying this condition and comparing the coefficients of the Taylor series expansions, it follows that $\zeta \circ \eta^{-1} = (\text{const.})z$, and the uniqueness follows. □

REMARK 3.3. The problem of linearizing non-hyperbolic germs is extremely difficult and has a long history. In fact, if $\nu = e^{2\pi i t}$ with $0 < t < 1$ irrational, it turns out that the possibility of linearizing a germ f with $f'(0) = \nu$ depends on the asymptotic behavior of the denominators of the continued fraction approximations to t. Part of the linearization problem was solved by A. Brjuno in the mid 60's, but the complete solution has been found only in recent years with the work of J.C. Yoccoz and R. Perez-Marco [**P**].

3.2. Approximation by Elements of a Pseudo-Group. The next proposition shows how to approximate a linear map in a suitable coordinate system by elements of a given pseudo-group PG of germs in a finitely-generated subgroup $G \subset \mathrm{Bih}_0(\mathbb{C})$. The proof is nothing but an elaboration of the following elementary fact: If $f \in \mathrm{Bih}_0(\mathbb{C})$ and $|\nu_1| < 1$, then $f'(0)z = \lim_{n\to\infty} \nu_1^{-n} f(\nu_1^n z)$.

PROPOSITION 3.4. *Let G be a marked subgroup of $\mathrm{Bih}_0(\mathbb{C})$ with generators $f_1, \ldots, f_k$, all defined on some domain Ω containing 0. Let f_1 be a hyperbolic germ and ζ be a holomorphic coordinate change linearizing f_1. Without loss of generality, assume that $|f_1'(0)| < 1$, ζ is defined on Ω, and $\zeta(\Omega) = \mathbb{D}(0, r)$ for some $r > 0$. Let DG be the tangent group of G, i.e., the multiplicative subgroup of $\mathbb{C}^*$ generated by $\nu_j := f_j'(0)$, $1 \leq j \leq k$. Then for every $\nu \in \overline{DG}$, there exists a sequence F_n in the pseudo-group PG which in coordinate ζ converges to the linear map $\zeta \mapsto \nu\zeta$ uniformly on compact subsets of $\{\zeta : |\zeta| < \min(r, r/|\nu|)\} = \zeta(\Omega \cap \nu^{-1}\Omega)$.*

By an abuse of notation, we denote by $f(\zeta)$ the germ induced by f in the coordinate ζ, where $f \in \mathrm{Bih}_0(\mathbb{C})$ and ζ is a holomorphic change of coordinate near 0.

Proof. It suffices to consider the case where $\nu \in DG$, i.e., when $\nu = f'(0)$ for some $f \in G$. The general case will then follow by the uniformity of convergence and a standard diagonal argument. Define

$$F_n := f_1^{-n} \circ f \circ f_1^n. \tag{3.2}$$

First we claim that F_n is defined on Ω for all sufficiently large n, and $F_n(\zeta) \to \nu\zeta$ uniformly on compact subsets of $\mathbb{D}(0,r)$ as $n \to \infty$. In fact, if $f(\zeta) = \nu\zeta + \sum_{j=2}^{\infty} a_j \zeta^j$, then it easily follows from (3.2) that

$$F_n(\zeta) = \nu\zeta + \sum_{j=2}^{\infty} a_j \nu_1^{n(j-1)} \zeta^j. \tag{3.3}$$

Since $f(\zeta)$ is holomorphic on $\mathbb{D}(0,r')$ for some $0 < r' < r$, one has $\limsup_j \sqrt[j]{|a_j|} \leq 1/r'$, so for large n, $\limsup_j \sqrt[j]{|a_j||\nu_1|^{n(j-1)}} \leq 1/r$. Thus the expression on the right hand side of (3.3) has an analytic continuation to $\mathbb{D}(0,r)$, i.e., F_n is defined in Ω for large n. Now the fact that $F_n(\zeta) \to \nu\zeta$ uniformly on compact subsets of $\mathbb{D}(0,r)$ is immediate from (3.3).

What remains to be shown is that for every compact set $K \subset \Omega \cap \nu^{-1}\Omega$ there is an $N = N(K) > 0$ such that the domain Ω_{F_n} of F_n as an element of PG (see Definition 2.12) contains K for all $n > N$.

Each intermediate representation of F_n in (3.2) has the form

$$g_m := f_1^m \quad \text{or} \quad h_{mn} := f_1^{-(n-m)} \circ f \circ f_1^n, \ 0 \leq m \leq n.$$

We shall prove that for large n, $g_m(\zeta)$ and $h_{mn}(\zeta)$ have conformal extensions to $\mathbb{D}(0,r)$ and $K' := \zeta(K)$ is mapped into $\mathbb{D}(0,r)$ by them. In fact, $g_m(\zeta) = \nu_1^m\zeta$ and $h_{mn}(\zeta) \to \nu_1^m\nu\zeta$ uniformly in ζ and m as $n \to \infty$, so that $g_m(\zeta)$ and $h_{mn}(\zeta)$ have conformal extensions over $\mathbb{D}(0,r)$ for large n. Now $g_m(\zeta(K)) = \nu_1^m K' \subset \mathbb{D}(0,r)$ for all $m \geq 0$. On the other hand, let $\delta := \sup\{|\zeta| : \zeta \in K'\}$, so that $\delta < \min(r, r/|\nu|)$. Choose $0 < \epsilon < r - |\nu|\delta$, and find $N > 0$ such that $|h_{mn}(\zeta) - \nu_1^m\nu\zeta| < \epsilon$ for all $0 \leq m \leq n$ and $\zeta \in \mathbb{D}(0,r)$ whenever $n > N$. Then if $\zeta \in K'$ we have $|h_{mn}(\zeta)| < \epsilon + |\nu_1|^m|\nu|\delta < r$ for $n > N$ and all $0 \leq m \leq n$. $\square$

Theorem 3.5. *Let $G \subset \mathrm{Bih}_0(\mathbb{C})$ be a marked subgroup, and suppose that the tangent group DG is dense in $\mathbb{C}$. Then there exists an open neighborhood Ω of 0 such that for every $z \in \Omega \smallsetminus \{0\}$ the orbit of z under the pseudo-group PG is dense in Ω.*

Proof. Since $\overline{DG} = \mathbb{C}$, G must contain at least one hyperbolic germ, say f_1. Let Ω and ζ be as in Proposition 3.4. By density of DG, for every $\nu \in \mathbb{C}$ there exists a sequence $\{F_n\}$ in PG such that $F_n(\zeta) \to \nu\zeta$ uniformly on compact subsets of $\zeta(\Omega \cap \nu^{-1}\Omega)$. Choose $z \in \Omega \smallsetminus \{0\}$ and let $w \in \Omega$ be arbitrary. Set $\nu := \zeta(w)/\zeta(z)$. Then $F_n(\zeta(z)) \to (\zeta(w)/\zeta(z))\ \zeta(z) = \zeta(w)$ as $n \to \infty$, and we are done. $\square$

3.3. Ergodicity in Subgroups of $\mathbf{Bih}_0(\mathbb{C})$.

Recall from ergodic theory that a measure-preserving transformation T acting on a probability space X is called *ergodic* if every T-invariant subset of X has measure 0 or 1. In what follows, for two measurable sets $A, B \subset \mathbb{C}$, the notation $A \doteq B$ means the symmetric difference $(A \smallsetminus B) \cup (B \smallsetminus A)$ has Lebesgue measure zero.

Definition 3.6. Two Lebesgue measurable subsets A and B of $\mathbb{C}$ are said to be *equivalent at* 0 if there exists an open disk U around 0 such that $A \cap U \doteq B \cap U$.

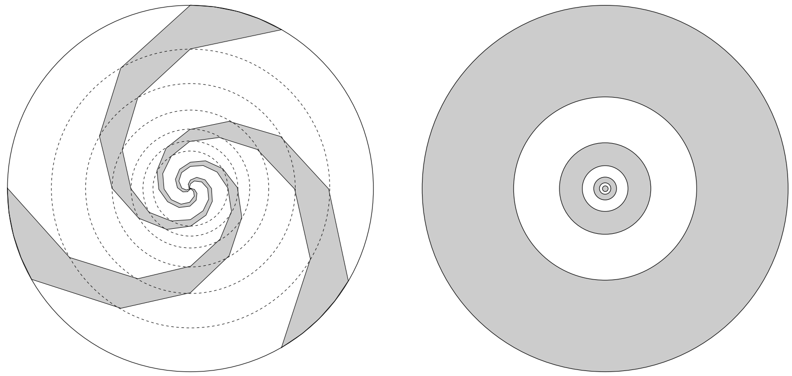

FIGURE 10. *Invariant sets A for the action $f : z \mapsto \nu z$ with $[A] \neq [\mathbb{C}]$ and $[A] \neq [\emptyset]$. Left: $|\nu| \neq 1$. Right: $|\nu| = 1$.*

The *germ* of A, denoted by $[A]$, is the equivalence class of A under this relation. Given an $f \in \mathrm{Bih}_0(\mathbb{C})$, a set A is called *f-invariant* if $[A] = [f(A)]$. Given a subgroup $G \subset \mathrm{Bih}_0(\mathbb{C})$, a set A is said to be *G-invariant* if it is f-invariant for every $f \in G$. The subgroup G is called *ergodic* if for every G-invariant set A, we have $[A] = [\mathbb{C}]$ or $[\emptyset]$.

EXAMPLE 3.7. Let $f(z) = \nu z$, $\nu \neq 0$, and G be the subgroup of $\mathrm{Bih}_0(\mathbb{C})$ generated by f. Then G is not ergodic since it is easy to construct an f-invariant measurable set A such that $[A] \neq [\mathbb{C}]$ and $[A] \neq [\emptyset]$. For example, if $|\nu| < 1$ (the case $|\nu| > 1$ is similar), take any sector B in the fundamental annulus $\{z : |\nu| < |z| \leq 1\}$ and set $A = \bigcup_{n=0}^{\infty} \nu^n B$ (see Fig. 10 left). If, on the other hand, $|\nu| = 1$, consider the invariant set $A = \bigcup_{n=0}^{\infty} \{z : 2^{-2n-1} \leq |z| \leq 2^{-2n}\}$ (see Fig. 10 right). However, we will see that if G is generated by two linear germs $f_1(z) = \nu_1 z$ and $f_2(z) = \nu_2 z$, with the tangent group $DG = \langle \nu_1, \nu_2 \rangle$ dense in $\mathbb{C}$, then G is ergodic (see Proposition 3.13).

DEFINITION 3.8. Let $A \subset \mathbb{C}$ be Lebesgue measurable. A point $z \in \mathbb{C}$ is called a *density point* of A if

$$\lim_{r \to 0} \frac{m(A \cap \mathbb{D}(z,r))}{m(\mathbb{D}(z,r))} = 1, \tag{3.4}$$

where m denotes Lebesgue measure on $\mathbb{C}$.

It is well-known that *almost every point of A is a density point of A* (see [**Ru**] for a proof).

Our next goal is to prove that under fairly general circumstances, a marked subgroup of $\mathrm{Bih}_0(\mathbb{C})$ is ergodic. The idea of the proof, due to E. Ghys, is based on a few preliminary statements, which we present below. The first one is the classical "Koebe 1/4-Theorem" (see [**Ru**]):

THEOREM 3.9. *Let $f : \mathbb{D}(z,r) \to \mathbb{C}$ be a univalent function. Then $f(\mathbb{D}(z,r))$ contains the disk $\mathbb{D}(f(z), |f'(z)|r/4)$.*

LEMMA 3.10. *Let $f(z) = \nu z$, with $0 < |\nu| < 1$, and $g \in \mathrm{Bih}_0(\mathbb{C})$. Suppose that A is both f- and g-invariant. Then there exists an open disk U around 0 such that $A \cap U \doteq \nu^{-n} g \nu^n(A) \cap U$ for all $n \geq 0$.*

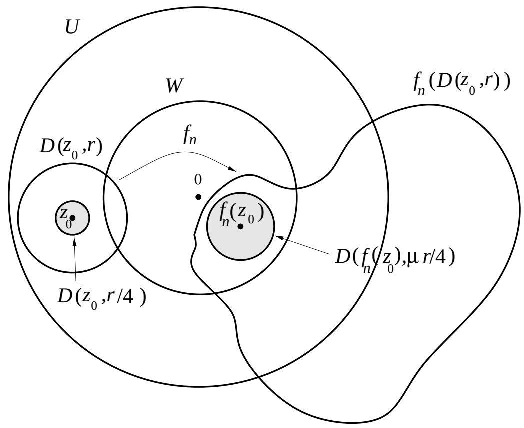

Figure 11. *Proof of Proposition 3.11.*

Proof. Evidently, for each integer n, A is invariant under $f^{-n}\circ g\circ f^n$. To prove the assertion, it suffices to find a disk U that works uniformly for all sufficiently large n. Without loss of generality, in what follows we replace $\doteq$ by $=$ by modifying the inclusions up to a set of measure zero.

Choose a disk $V = \mathbb{D}(0, r_1)$ such that

$$A\cap V = (\nu A)\cap V = (\nu^{-1}A)\cap V = g(A)\cap V. \tag{3.5}$$

Set $\mu := 1/g'(0)$. Since $\nu^{-n}g^{-1}\nu^n$ converges uniformly to $z\mapsto \mu z$ in a neighborhood of 0, we can choose a disk $U = \mathbb{D}(0, r_2)\subset V$ and an $N > 0$ such that $|\nu^{-n}g^{-1}(\nu^n z)| \le |\mu||z| + r_1/2$ for all $z\in U$ and all $n > N$. Moreover, we may choose r_2 such that $|\mu|r_2 < r_1/2$ and N large enough such that $\nu^n U\subset g(U)$ for all $n > N$.

Now suppose that $z\in A\cap U$. Then by (3.5), $w = \nu^n z\in A\cap U$. Again by (3.5) we have $w = g(x)$, where $x\in A$. If $n > N$, x is in fact in $A\cap U$ by the choice of N. If $y = \nu^{-n}x$, then $|y| = |\nu^{-n}g^{-1}(\nu^n z)| \le |\mu||z| + r_1/2 \le |\mu|r_2 + r_1/2 < r_1$, so that $y\in V$. Once again by (3.5) we obtain $y\in A\cap V$. Hence, $z\in \nu^{-n}g\nu^n(A)\cap U$.

The proof that $\nu^{-n}g\nu^n(A)\cap U\subset A\cap U$ for all large n is quite similar. □

Proposition 3.11. *Let $f_n\in \mathrm{Bih}_0(\mathbb{C})$ be defined on some open neighborhood V of 0 for all $n\ge 1$, and $f_n\to f\in \mathrm{Bih}_0(\mathbb{C})$ uniformly on compact subsets of V. Suppose that A is f_n-invariant and that there exists a disk $U\subset V$ around 0 such that $A\cap U\doteq f_n(A)\cap U$ for all $n\ge 1$. Then A is f-invariant.*

Proof. Without loss of generality we may assume that A has positive measure and each point of A is a density point. Choose a smaller disk $W\subset U\cap f(U)\cap f^{-1}(U)$ such that $f_n(W)\subset U$ for all n. Choose $z_0\in A$ such that $f(z_0)\in W$. We show that $f(z_0)$ is a density point for A. This proves $f(A)\cap W\subset A\cap W$. Next, by the choice of W, $f_n^{-1}(A)\cap W = A\cap W$, so the same argument gives us $A\cap W\subset f(A)\cap W$.

So let $z_0\in A$ with $f(z_0)\in W$. By the choice of W, we have $z_0\in U$. Given any $\epsilon > 0$, choose $r = r(\epsilon) > 0$ so small that

(i) $\mathbb{D}(z_0, r)\subset U$,

(ii) $1-\epsilon < \dfrac{m(A\cap \mathbb{D}(z_0, r'))}{m(\mathbb{D}(z_0, r'))} \le 1$ for every $0 < r' \le r$,

(iii) $|f'(z_0)| - \epsilon \leq \inf_{z \in \mathbb{D}(z_0,r)} |f'_n(z)| \leq \sup_{z \in \mathbb{D}(z_0,r)} |f'_n(z)| \leq |f'(z_0)| + \epsilon$ for all large n.

For simplicity, set $\mu := |f'(z_0)| - \epsilon$, $D := \mathbb{D}(z_0, r)$, $D' := \mathbb{D}(f(z_0), \mu r/4)$, and $D'_n := \mathbb{D}(f_n(z_0), \mu r/4)$. By Theorem 3.9, $f_n(D)$ contains D'_n if n is large enough (Fig. 11). Since $D'_n \smallsetminus A \subset f_n(D) \smallsetminus (A \cap f_n(D)) = f_n(D) \smallsetminus f_n(A \cap D)$, for large n we have

$$\begin{aligned} m(A \cap D'_n) &\geq m(D'_n) - m(f_n(D)) + m(f_n(A \cap D)) \\ &= m(D'_n) - \int_D |f'_n|^2 \, dm + \int_{A \cap D} |f'_n|^2 \, dm \\ &\geq m(D'_n) - (|f'(z_0)| + \epsilon)^2 \, m(D) + \mu^2 \, m(A \cap D) && \text{(by (iii))} \\ &\geq [1 - 16(|f'(z_0)| + \epsilon)^2 \mu^{-2} + 16(1-\epsilon)] \, m(D'_n) && \text{(by (ii))} \\ &=: \ell(\epsilon) \, m(D'_n). \end{aligned}$$

As $n \to \infty$, $m(A \cap D'_n) \to m(A \cap D')$ by Lebesgue's Dominated Convergence Theorem. Therefore

$$\frac{m(A \cap D')}{m(D')} \geq \ell(\epsilon).$$

Since $\ell(\epsilon) \to 1$ as $\epsilon \to 0$, it follows that $f(z_0)$ is a density point of A. □

Corollary 3.12. *Under the assumptions of Proposition 3.4, every G-invariant set A is also DG-invariant.*

Proof. Let $\nu = g'(0) \in DG$ for some $g \in G$. In the coordinate ζ one has $f_1(\zeta) = \nu_1 \zeta$, and by Proposition 3.4, $f_1^{-n} \circ g \circ f_1^n$ converges to $\zeta \to \nu\zeta$ uniformly on compact subsets of $\zeta(\Omega \cap \nu^{-1}\Omega)$. It follows from Lemma 3.10 and Proposition 3.11 that A is invariant under $\zeta \mapsto \nu\zeta$. □

The proof of the following proposition uses a standard technique in ergodic theory (see for example Appendix 11 of [**AA**]).

Proposition 3.13. *Choose $\nu_j \in \mathbb{C}^*$, $1 \leq j \leq k$, and let G be the subgroup of* $\mathrm{Bih}_0(\mathbb{C})$ *generated by the linear germs $z \mapsto \nu_j z$, $1 \leq j \leq k$. Then G is ergodic if and only if its tangent group DG is dense in $\mathbb{C}$.*

Proof. Suppose that DG is dense in $\mathbb{C}$. Then G contains at least one hyperbolic element $f(z) = \nu z$ with $|\nu| < 1$. Let $A \subset \mathbb{C}$ be any measurable G-invariant set. Choose a disk U around 0 such that $A \cap U \doteq (\nu A) \cap U$. Take the quotient of $U \smallsetminus \{0\}$ under the action of the group $\{f^n : n \in \mathbb{Z}\}$ generated by f, which is biholomorphic to a 2-torus $\mathbb{T}^2$. Let $\tilde{G}$ be the induced group of translations of $\mathbb{T}^2$, and $\tilde{A}$ be the induced measurable subset of $\mathbb{T}^2$. Note that $\tilde{A}$ is invariant under the action of $\tilde{G}$, and the orbit of each point in $\mathbb{T}^2$ is dense under this action.

It suffices to show that $\tilde{A}$ or $\mathbb{T}^2 \smallsetminus \tilde{A}$ has measure zero. Expand the characteristic function of $\tilde{A}$ into the Fourier series

$$\chi_{\tilde{A}}(e^{2\pi i x}, e^{2\pi i y}) = \sum_{m,n} a_{mn} \, e^{2\pi i (mx + ny)},$$

where we identify $\mathbb{T}^2$ with $\{(e^{2\pi i x}, e^{2\pi i y}) \in S^1 \times S^1\}$. Let $\tilde{f} \in \tilde{G}$ be the translation $(e^{2\pi i x}, e^{2\pi i y}) \mapsto (e^{2\pi i (x+\alpha)}, e^{2\pi i (y+\beta)})$. The $\tilde{G}$-invariance of $\tilde{A}$ shows that

$$\chi_{\tilde{A}} = \sum_{m,n} a_{mn} e^{2\pi i (m\alpha + n\beta)} \, e^{2\pi i (mx + ny)}$$

almost everywhere. Therefore, for all $m, n \in \mathbb{Z}$, $a_{mn} = a_{mn}\, e^{2\pi i(m\alpha+n\beta)}$. Since $\tilde{G}$ contains at least one irrational translation (otherwise, the orbit of each point would be finite), we conclude that $a_{mn} = 0$ for all $(m,n) \neq (0,0)$, hence $\chi_{\tilde{A}} = 0$ or 1 almost everywhere.

Conversely, let G be ergodic. Clearly G contains a hyperbolic element $f_0(z) = \nu z$, with $|\nu| < 1$. Suppose by way of contradiction that DG is not dense in $\mathbb{C}$, so that there exists an open disk $\mathbb{D}(z,r)$ such that $\overline{DG} \cap \mathbb{D}(z,r) = \emptyset$. Set $A := \bigcup_{f\in G} f(\mathbb{D}(z,r/2))$. Then A is an open, G-invariant set such that $\overline{DG} \cap A = \emptyset$. The germ of A at 0, denoted by $[A]$, is not equal to $[\emptyset]$ since every neighborhood of 0 contains $\nu^n \mathbb{D}(z,r/2)$ for large n. It follows that $[A] = [\mathbb{C}]$. The invariance of A and the fact that $|\nu| < 1$ will then show that $m(\mathbb{C} \smallsetminus A) = 0$. In particular, A is dense in $\mathbb{C}$. Let $\{w_n\}$ be a sequence in A such that $\lim w_n$=1. Let $w_n = \sigma_n z_n$, where $\sigma_n \in DG$ and $z_n \in \mathbb{D}(z,r/2)$. Then σ_n^{-1} belongs to $\mathbb{D}(z,r)$ for large n . This contradicts the assumption $\overline{DG} \cap \mathbb{D}(z,r) = \emptyset$. ☐

REMARK 3.14. The above proof shows how the notion of ergodicity for finitely-generated subgroups of $\text{Bih}_0(\mathbb{C})$ containing a hyperbolic germ is related to the usual notion of ergodicity for translations of tori, justifying Definition 3.6.

THEOREM 3.15. *Let $G \subset \text{Bih}_0(\mathbb{C})$ be a marked subgroup, with the tangent group DG dense in $\mathbb{C}$. Then G is ergodic.*

PROOF. Since DG is dense in $\mathbb{C}$, G must contain at least one hyperbolic element. By Corollary 3.12, every G-invariant set A is also DG-invariant. By Proposition 3.13, $[A] = [\mathbb{C}]$ or $[\emptyset]$. ☐

3.4. Density of Leaves of SHFC's on $\mathbb{CP}^2$. This section is devoted to the proof of a theorem of Khudai-Veronov on density of leaves of a typical SHFC on $\mathbb{CP}^2$. We first prove a version of this theorem which asserts that for a typical $\mathcal{F} \in \mathcal{A}_n$ all but a finite number of leaves are dense in $\mathbb{CP}^2$. Next, by applying a more elaborate argument, we show that among these exceptional leaves only $\mathcal{L}_\infty$ is robust. In fact, for a typical $\mathcal{F} \in \mathcal{A}_n$ *all* leaves except $\mathcal{L}_\infty$ are dense in $\mathbb{CP}^2$.

DEFINITION 3.16. Let X be a holomorphic vector field defined in some neighborhood U of $p \in \mathbb{C}^2$, and let p be an isolated singular point of X. Let σ_1 and σ_2 be the eigenvalues of the Jacobian matrix $DX(p)$. We say that p is a *non-degenerate* singularity if $\sigma_1\sigma_2 \neq 0$. p is called a *hyperbolic* singularity if it is non-degenerate and the characteristic number σ_1/σ_2 is not real.

THEOREM 3.17. *Every hyperbolic singularity is locally linearizable: There exist neighborhoods U of p and V of $0 \in \mathbb{C}^2$ and a biholomorphism $\varphi : U \xrightarrow{\simeq} V$ such that $(\varphi_* X)(x,y) = \sigma_1 x\, \partial/\partial x + \sigma_2 y\, \partial/\partial y$.*

In fact, in this two dimensional case, the linearization is possible when either $\sigma_1/\sigma_2 \notin \mathbb{R}$, or σ_1/σ_2 is positive but not an integer or the inverse of an integer (the so-called "non-resonant Poincaré case"). For a proof, see [**Ar**].

Using the above theorem, we can understand the local picture of a hyperbolic singularity since it is easy to integrate the linearized vector field.

Recall that for a holomorphic vector field X with an isolated singularity at p, a *local separatrix* through p is the image of a punctured disk $\mathbb{D}^*(0,r)$ under a holomorphic immersion η such that $d\eta(T)/dT = X(\eta(T))$ for $T \in \mathbb{D}^*(0,r)$, and $\lim_{T\to 0} \eta(T) = p$. It follows that a local separatrix of p is an invariant analytic

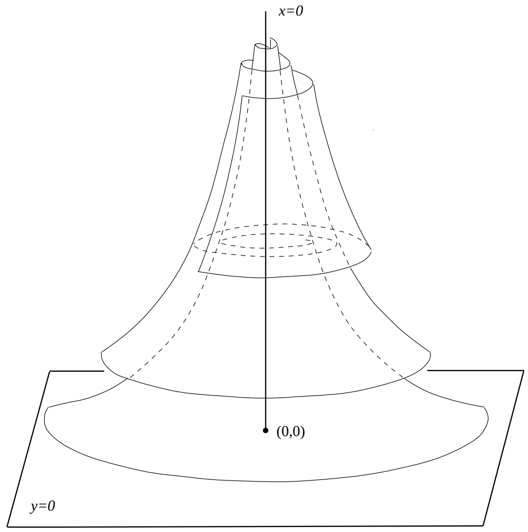

FIGURE 12. *A hyperbolic singularity at the origin of* $\mathbb{C}^2$.

curve which passes through p, perhaps with a singularity there. It is shown by C. Camacho and P. Sad that every holomorphic vector field X with an isolated singularity at $0 \in \mathbb{C}^2$ has a local separatrix through 0 [**CS**]. On the other hand, this statement is false in higher dimensions, even in $\mathbb{C}^3$ [**GL**].

COROLLARY 3.18. *Let X be a holomorphic vector field defined in a neighborhood of an isolated hyperbolic singularity $p \in \mathbb{C}^2$. Then X has exactly two local separatrices through p, and every other integral curve near p accumulates on these two separatrices.*

Fig. 12 is an attempt to visualize this situation.

PROOF. By Theorem 3.17 X can locally be transformed into $\sigma_1 x\ \partial/\partial x + \sigma_2 y\ \partial/\partial y$. The integral curve passing through (x_0, y_0) can be parametrized as $T \mapsto (x_0 e^{\sigma_1 T}, y_0 e^{\sigma_2 T})$. It follows that the punctured axes $\{x = 0\} \smallsetminus \{(0,0)\}$ and $\{y = 0\} \smallsetminus \{(0,0)\}$ are local separatrices. Since $\sigma_1/\sigma_2 \notin \mathbb{R}$, there exist sequences $\{T_n\}$ and $\{T'_n\}$ such that

$$\begin{aligned} e^{\sigma_1 T_n} &= 1 \text{ for } n = 1, 2, \dots, & e^{\sigma_2 T_n} &\to 0 \text{ as } n \to \infty, \\ e^{\sigma_2 T'_n} &= 1 \text{ for } n = 1, 2, \dots, & e^{\sigma_1 T'_n} &\to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows that if $x_0 y_0 \neq 0$, the integral curve passing through (x_0, y_0) accumulates on $(x_0, 0)$ and $(0, y_0)$, hence on the axes $\{x = 0\}$ and $\{y = 0\}$ by Proposition 1.6. $\square$

Now let $\mathcal{F} \in \mathcal{A}'_n$, $L_0 \cap \operatorname{sing}(\mathcal{F}) = \{p_1, \dots, p_{n+1}\}$, and let λ_j be the characteristic number of p_j, as in (2.7). It follows that p_j is a hyperbolic singularity of X_1 (hence of any vector field representing $\mathcal{F}$ near p_j) if and only if $\lambda_j \notin \mathbb{R}$. By Proposition 2.19, the last condition is equivalent to $|\nu_j| \neq 1$, where ν_j is the multiplier at 0 of the monodromy mapping $f_j \in G_\infty$. We conclude that p_j *is a hyperbolic singularity if and only if* f_j *is a hyperbolic germ in* $\mathrm{Bih}_0(\mathbb{C})$.

Note that if p_j is hyperbolic, then one of the separatrices through p_j is the leaf at infinity; the other one is transversal to $\mathcal{L}_\infty$ (Fig. 13).

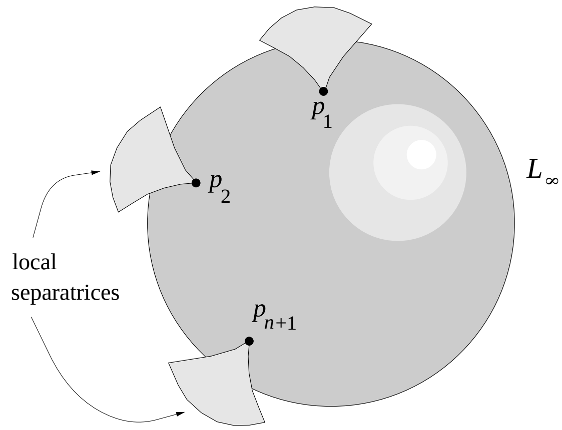

FIGURE 13. *Leaf at infinity with hyperbolic singularities.*

PROPOSITION 3.19. *A typical $\mathcal{F} \in \mathcal{A}_n$, $n \geq 2$, satisfies the following conditions:*

(i) $\mathcal{F} \in \mathcal{A}'_n$,
(ii) $|\nu_j| \neq 1$ *for all* $1 \leq j \leq n$*; in other words, all the generators of* G_∞ *are hyperbolic,*
(iii) *the tangent group* $DG_\infty = \langle \nu_1, \ldots, \nu_n \rangle$ *is dense in* $\mathbb{C}$.

PROOF. Since the union of sets of measure zero has measure zero, it suffices to prove that each condition is typical in $\mathcal{A}_n$.

The first condition is typical by Corollary 1.29. The second one is typical by (2.7): If $|\nu_j| = 1$, then $\lambda_j \in \mathbb{R}$, and this can be easily destroyed by perturbing the coefficients of P and Q.

The third condition is more subtle. By (2.7) and Proposition 2.19, it suffices to prove that for almost every $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, the multiplicative subgroup generated by $\{e^{2\pi i \lambda_1}, \ldots, e^{2\pi i \lambda_n}\}$ is dense in $\mathbb{C}$. Evidently it is enough to prove this statement for $n = 2$, since then we can take the product of the resulting subset of $\mathbb{C}^2$ by $\mathbb{C}^{n-2}$ to obtain a subset of full measure in $\mathbb{C}^n$.

We shall prove that for almost every $(\lambda_1, \lambda_2) \in \mathbb{C}^2$, the additive subgroup generated by $\{1, \lambda_1, \lambda_2\}$ is dense in $\mathbb{C}$. Suppose that (λ_1, λ_2) is chosen such that

($*$) no two vectors in $\{1, \lambda_1, \lambda_2\}$ are $\mathbb{R}$-dependent,
($**$) if $1 = a\lambda_1 + b\lambda_2$, with $a, b \in \mathbb{R}$, then b/a is irrational.

Let Λ be the lattice generated by $\{\lambda_1, \lambda_2\}$, and consider the quotient torus $\mathbb{R}^2 \xrightarrow{\pi} \mathbb{R}^2/\Lambda$. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with $L(\lambda_1) = (1, 0)$, $L(\lambda_2) = (0, 1)$. Then $L(1) = (a, b)$, and L induces a homeomorphism $\tilde{L} : \mathbb{R}^2/\Lambda \to \mathbb{R}^2/\mathbb{Z}^2$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{L} & \mathbb{R}^2 \\ \Big\downarrow \pi & & \Big\downarrow \pi' \\ \mathbb{R}^2/\Lambda & \xrightarrow{\tilde{L}} & \mathbb{R}^2/\mathbb{Z}^2 \end{array} \tag{3.6}$$

Note that the slope of $L(1)$ is irrational by ($**$). So the sequence $\{\pi'(L(n))\}_{n \geq 0}$ is dense in $\mathbb{R}^2/\mathbb{Z}^2$. Pulling back this sequence to the torus $\mathbb{R}^2/\Lambda$ by $\tilde{L}$, it follows from diagram (3.6) that the sequence $\{\pi(n)\}_{n\geq 0}$ is dense in $\mathbb{R}^2/\Lambda$. Hence if (λ_1, λ_2) satisfies $(*)$ and $(**)$, the subgroup generated by $\{1, \lambda_1, \lambda_2\}$ is dense in $\mathbb{C}$. Finally, it is straightforward to check that $(*)$ and $(**)$ hold for almost every $(\lambda_1, \lambda_2) \in \mathbb{C}^2$. $\square$

Now the first version of the density theorem is quite easy to prove.

THEOREM 3.20. *For a typical $\mathcal{F} \in \mathcal{A}_n$, all but at most $n+2$ non-singular leaves are dense in $\mathbb{CP}^2$.*

PROOF. Let $\mathcal{F} \in \mathcal{A}_n$ have properties (i), (ii), and (iii) of Proposition 3.19. Since all the singular points $\{p_1, \ldots, p_{n+1}\} = L_0 \cap \operatorname{sing}(\mathcal{F})$ are hyperbolic, there are exactly two local separatrices through each p_j, one of them lies in $\mathcal{L}_\infty$. Denote by $\mathcal{L}_j$ the *global separatrix* through p_j which is transversal to $\mathcal{L}_\infty$. (By the global separatrix we mean the extension of the local separatrix as a leaf.) Note that we might have $\mathcal{L}_i = \mathcal{L}_j$ even if $i \neq j$.

Now let $p \in \mathbb{CP}^2 \smallsetminus (\operatorname{sing}(\mathcal{F}) \cup \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_{n+1} \cup \mathcal{L}_\infty)$. By Corollary 2.17, $\mathcal{L}_p$ has a point of accumulation on L_0. By the choice of p and Corollary 3.18, this point can be chosen on $\mathcal{L}_\infty$, hence by Proposition 1.6 the entire $\mathcal{L}_\infty$ is in the closure of $\mathcal{L}_p$. In particular, $\mathcal{L}_p$ intersects the transversal Σ to $\mathcal{L}_\infty$ at the base point a of $\pi_1(\mathcal{L}_\infty)$. By Theorem 3.5, $\mathcal{L}_p \cap \Sigma$ is dense in some neighborhood $\Omega \subset \Sigma$ of a.

Now choose any open set $U \subset \mathbb{CP}^2$ and any point $q \in U \smallsetminus (\operatorname{sing}(\mathcal{F}) \cup \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_{n+1} \cup \mathcal{L}_\infty)$. By the above argument, $\mathcal{L}_q \cap \Sigma$ is dense in Ω. Therefore $q \in \overline{\mathcal{L}_p}$ by another application of Proposition 1.6. Since U was arbitrary, $\mathcal{L}_p$ is dense in $\mathbb{CP}^2$. $\square$

The above proof shows that all non-singular leaves other than $\mathcal{L}_\infty$ and the global separatrices $\mathcal{L}_j$ are dense. The condition $p \notin \bigcup_{j=1}^{n+1} \mathcal{L}_j$ is only used to guarantee the existence of point in $\mathcal{L}_\infty \cap \overline{\mathcal{L}_p}$. If the only accumulation points of $\mathcal{L}_p$ on L_0 are singular, it turns out that $\overline{\mathcal{L}_p}$ is an algebraic curve. Since the algebraic leaves other than $\mathcal{L}_\infty$ are not typical for elements of $\mathcal{A}_n$ (Proposition 3.21 below), we can prove a stronger version of the density theorem without the assumption $p \notin \bigcup_{j=1}^{n+1} \mathcal{L}_j$ in the above proof.

PROPOSITION 3.21. *For a typical $\mathcal{F} \in \mathcal{A}_n$, the only algebraic leaf is the leaf at infinity $\mathcal{L}_\infty$.*

PROOF. The argument below is an adaptation of an idea due to I. Petrovskiĭ and E. Landis [**PL1**]. Let $\mathcal{F} : \{P dy - Q dx = 0\} \in \mathcal{A}_n$. Suppose that the algebraic curve $S_K : \{K = 0\}$ (see §1.4) with singular points of $\mathcal{F}$ deleted is a leaf of $\mathcal{F}$, where $K = K(x, y)$ is an irreducible polynomial of degree k. Since S_K is a leaf, we have

$$\frac{\partial K}{\partial x}(x,y)P(x,y) + \frac{\partial K}{\partial y}(x,y)Q(x,y) = 0$$

whenever $K(x, y) = 0$. It follows that there exists a polynomial $\tilde{K}$ of degree at most $(n - 1)$ such that

$$\frac{\partial K}{\partial x}P + \frac{\partial K}{\partial y}Q = K\tilde{K}. \tag{3.7}$$

Conversely, if there exist polynomials $K, \tilde{K}, P$, and Q satisfying (3.7) with K irreducible and P and Q relatively prime, then $S_K \smallsetminus \operatorname{sing}(\mathcal{F})$ is an algebraic leaf of $\mathcal{F} : \{P dy - Q dx = 0\}$.

Let E be the complex linear space of the coefficients of $K, \tilde{K}, P$, and Q, which has dimension

$$\frac{(k+1)(k+2)}{2} + \frac{n(n+1)}{2} + 2\ \frac{(n+1)(n+2)}{2} = \frac{1}{2}\{(k+1)(k+2) + (3n^2+7n+4)\}.$$

If we impose (3.7) on these coefficients, we obtain equalities that define an algebraic variety S in E. Note that if a belongs to S, so does λa for every $\lambda \in \mathbb{C}^*$ by (3.7). Therefore, S projects to an algebraic variety S^* in $\mathbb{CP}^d$, with $d = \dim E - 1$. Decompose S^* as $\bigcup_{j=1}^m S_j^*$, where each S_j^* is irreducible. Let $E' \subset E$ be the subspace of coefficients of P and Q, which has dimension n^2+3n+2. The linear projection $E \to E'$ induces a projection $\pi : \mathbb{CP}^d \to \mathbb{CP}^N$, where $N = (\dim E')-1 = n^2+3n+1$. Each $\pi(S_j^*)$ is an algebraic variety in $\mathbb{CP}^N$. Since there are SHFC's in $\mathcal{A}_n$ which do not have any algebraic leaf other than $\mathcal{L}_\infty$, we have $\pi(S_j^*) \neq \mathbb{CP}^N$, so $\dim \pi(S_j^*) \leq N-1$. Taking the union for all $j = 1, \cdots, m$, we obtain the algebraic variety $\pi(S^*)$ in $\mathbb{CP}^N$, each irreducible component of which has dimension $\leq N-1$. It follows that the Lebesgue measure of $\pi(S^*)$ in $\mathcal{A}_n$ is zero. Since points in $\pi(S^*) \cap \mathcal{A}_n$ correspond to SHFC's which have an algebraic leaf other than $\mathcal{L}_\infty$, we obtain the result. □

Remark 3.22. By a much more difficult argument, using an index theorem of Camacho and Sad and the concept of the Milnor number of a local branch of a singular point, A. Lins Neto has shown that for $n \geq 2$ there exists an open and dense subset of $\mathcal{D}_n$ (see Corollary 1.19) consisting of SHFC's which do not have any algebraic leaf [**L**].

Proposition 3.23. *Let $\mathcal{F} \in \mathcal{A}'_n$ and all points in $L_0 \cap \operatorname{sing}(\mathcal{F})$ be hyperbolic. Let $\mathcal{L}$ be a non-singular leaf of $\mathcal{F}$ such that $\overline{\mathcal{L}} \cap L_0$ consists of singular points only. Then $\mathcal{L}$ is an algebraic leaf.*

Proof. Let $\Omega = \mathbb{CP}^2 \smallsetminus \operatorname{sing}(\mathcal{F})$. First we show that $\mathcal{L}$ is closed in Ω. Since $\overline{\mathcal{L}} \cap L_0 \subset L_0 \cap \operatorname{sing}(\mathcal{F})$, it follows that $\mathcal{L}$ coincides with the global separatrix $\mathcal{L}_j$ which is transversal to $\mathcal{L}_\infty$ at the singular point p_j on L_0. Let $p \in \overline{\mathcal{L}}$ be any non-singular point. Then, by Proposition 1.6, $L_0 \cap \overline{\mathcal{L}}_p \subset L_0 \cap \overline{\mathcal{L}} \subset L_0 \cap \operatorname{sing}(\mathcal{L})$, which shows $\mathcal{L}_p$ also coincides with $\mathcal{L}_j$. Hence $\mathcal{L}_p = \mathcal{L}$ and $p \in \mathcal{L}$.

Next we show that $\overline{\mathcal{L}}$ is an analytic subvariety of $\mathbb{CP}^2$. Let $p \in \Omega$. If $p \in \overline{\mathcal{L}}$, then $p \in \mathcal{L}$ by the above argument. Suppose that (U, φ) is a foliation chart around p, Σ is a transversal to $\mathcal{L}_p = \mathcal{L}$ at p, Σ' is another transversal to $\mathcal{L}$ at p' near $p_j \in L_0 \cap \operatorname{sing}(\mathcal{F})$, and γ is any path in $\mathcal{L}$ joining p to p' (Fig. 14). Let $f_\gamma : \Sigma \to \Sigma'$ be the associated holonomy mapping. If there exists a sequence $p_n \in \mathcal{L} \cap \Sigma$ which converges to p, then by considering the sequence $f_\gamma(p_n) \in \mathcal{L} \cap \Sigma'$ we conclude from Corollary 3.18 that $\mathcal{L}$ must have a non-singular accumulation point on $\mathcal{L}_\infty$, which contradicts our assumption. Therefore, by choosing U small enough, the only plaque of $\mathcal{L}$ in U is the one which passes through p, and evidently there exists a holomorphic function $f : U \to \mathbb{C}$ such that $f^{-1}(0) = \mathcal{L} \cap U$.

This means $\overline{\mathcal{L}} \smallsetminus \operatorname{sing}(\mathcal{F})$ is a 1-dimensional analytic subvariety of Ω. Since $\dim(\operatorname{sing}(\mathcal{F})) = 0 < 1 = \dim(\overline{\mathcal{L}} \smallsetminus \operatorname{sing}(\mathcal{F}))$, the theorem of Remmert-Stein (see for example [**GR**]) shows that $\overline{\mathcal{L}}$ is an analytic subvariety of $\mathbb{CP}^2$. Finally, every analytic subvariety of $\mathbb{CP}^2$ is algebraic by Chow's Theorem [**GH**]. □

Now, by Theorem 3.20, Proposition 3.21 and Proposition 3.23, we obtain the density theorem of Khudai-Veronov:

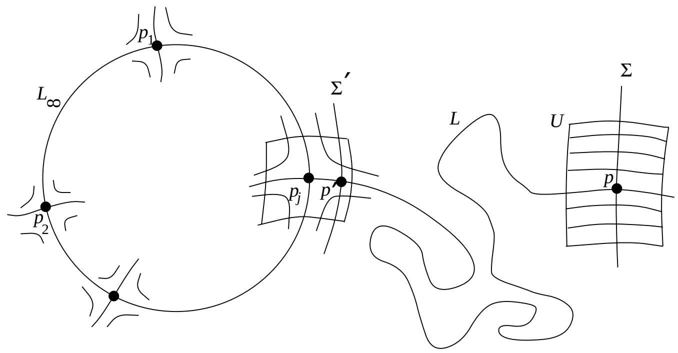

FIGURE 14. *Proof of Proposition 3.23.*

THEOREM 3.24. *For a typical $\mathcal{F} \in \mathcal{A}_n$, every leaf, except the leaf at infinity, is dense in $\mathbb{CP}^2$.*

3.5. Ergodicity of SHFC's on $\mathbb{CP}^2$. In what follows, we say that a set $A \subset \mathbb{CP}^2$ has measure zero if for every chart (U, φ) compatible with the standard smooth structure of $\mathbb{CP}^2$, the set $\varphi(A \cap U) \subset \mathbb{R}^4$ has Lebesgue measure zero. In other words, we consider the *Lebesgue measure class* on $\mathbb{CP}^2$.

DEFINITION 3.25. Let $\mathcal{F} \in \mathcal{A}_n$. The *saturation* $s(A)$ of $A \subset \mathbb{CP}^2$ is the set of all points q such that $q \in \mathcal{L}_p$ for some $p \in A$. A measurable set A is called *saturated* if $s(A) = A$ up to a set of measure zero. An SHFC $\mathcal{F}$ is called *ergodic* if for every measurable saturated set A, either A or $\mathbb{CP}^2 \smallsetminus A$ has measure zero.

Evidently if $\mathcal{F}$ is ergodic, then every non-singular leaf of $\mathcal{F}$ is either dense in $\mathbb{CP}^2$ or its closure has measure zero. Also the following observation is useful: Let $\mathcal{L}_p$ be a non-singular leaf and Σ be a transversal to $\mathcal{L}_p$ at p. Suppose that $A \subset \Sigma$ is a set of measure zero with respect to the Lebesgue measure class on $\Sigma \simeq \mathbb{D}$. Then $s(A)$ has measure zero in $\mathbb{CP}^2$. This is because $s(A)$ can be covered by a countable number of foliation charts $\{(U_i, \varphi_i)\}_{i=1}^{\infty}$ and each $s(A) \cap U_i$ has measure zero. In particular, each single leaf has measure zero (take $A = \{p\}$).

THEOREM 3.26. *Let $\mathcal{F} \in \mathcal{A}_n$ have properties (i),(ii), and (iii) of Proposition 3.19. Then $\mathcal{F}$ is ergodic. In particular, ergodicity is typical for elements of $\mathcal{A}_n$.*

PROOF. Let A be a measurable saturated subset of $\mathbb{CP}^2$. Without loss of generality, we may assume that A does not contain any global separatrix $\mathcal{L}_j$ through $p_j \in L_0 \cap \operatorname{sing}(\mathcal{F})$, for each individual leaf has measure zero. Let $p \in A$ be non-singular. Then $\mathcal{L}_p$ must accumulate on $\mathcal{L}_\infty$, so it has to intersect the transversal Σ to $\mathcal{L}_\infty$ at the base point a. By Theorem 3.15, G_∞ is ergodic. Since A is saturated, $A \cap \Sigma$ is G_∞-invariant, so there is an open disk $U \subset \Sigma$ around a such that either $A \cap U$ or $U \smallsetminus A$ has measure zero with respect to the Lebesgue measure class on Σ. It is clear that

$$s(A \cap U) = A \smallsetminus \operatorname{sing}(\mathcal{F}) \quad \text{and} \quad s(U \smallsetminus A) = \mathbb{CP}^2 \smallsetminus \{A \cup \operatorname{sing}(\mathcal{F})\}$$

up to a set of measure zero in $\mathbb{CP}^2$. By the observation before the statement of the theorem, it follows that either A or $\mathbb{CP}^2 \smallsetminus A$ has measure zero. $\square$

4. Non-Trivial Minimal Sets

This chapter deals with a somewhat different global aspect of SHFC's on $\mathbb{CP}^2$. As will be seen, the foliations under consideration are essentially those which do not have any algebraic leaf. In particular, because of the absence of the leaf at infinity, we cannot utilize such powerful tools as the monodromy group G_∞. Recall that a typical $\mathcal{F} \in \mathcal{A}_n$ has at least one algebraic leaf (i.e., $\mathcal{L}_\infty$). Hence from the point of view of differential equations for which the decomposition into the $\mathcal{A}_n$ is more natural, the foliations we consider in this chapter almost never occur. However, from the point of view of foliation theory, for which the natural decomposition is by the $\mathcal{D}_n$, the property of having no algebraic leaf seems to be typical (see Remark 3.22).

4.1. An Open Problem. The study of limit sets of foliations and flows in the real domain has proved to be of great significance in those theories. The classical theorem of Poincaré-Bendixson asserts that for every smooth real flow on the 2-sphere, every trajectory accumulates either on a periodic orbit or a singular point (or both). It is natural to ask a similar question for SHFC's on $\mathbb{CP}^2$. Here the analogue of a periodic orbit is a compact non-singular leaf and it is not difficult to prove that no such leaves could exist (Theorem 4.10). So we naturally arrive at the following question, apparently first asked by C. Camacho:

Question. *Is there a non-singular leaf of an SHFC on* $\mathbb{CP}^2$ *which does not accumulate on any singular point?*

Oddly enough, the question has remained open since the mid 80's. One can formulate it in a slightly different language, commonly used in foliation theory.

Definition 4.1. A *minimal set* for an SHFC on $\mathbb{CP}^2$ is a compact saturated non-empty subset of $\mathbb{CP}^2$ which is minimal with respect to these three properties. A *non-trivial minimal set* is a minimal set which is not a singular point. Throughout this chapter, $\mathcal{M}$ will always denote a non-trivial minimal set.

Minimality shows that if $p \in \mathcal{M}$, then $\overline{\mathcal{L}}_p = \mathcal{M}$. It follows that the problem of finding a non-singular leaf which does not accumulate on any singular point is equivalent to finding a non-trivial minimal set. Therefore, we can reformulate the above question as

The Minimal Set Problem. *Does there exist an SHFC on* $\mathbb{CP}^2$ *which has a non-trivial minimal set?*

Such a non-trivial minimal set is an example of a *Riemann surface lamination.* By definition, a Riemann surface lamination (RSL) is a compact space which locally looks like the product of the unit disk and a compact metric space (usually a Cantor set). The transition maps between various charts are required to be holomorphic in the leaf direction and only continuous in the transverse direction. Clearly every compact Riemann surface is such a space, but they form the class of *trivial* RSL's. Although there are some basic results on uniformization of RSL's (see [**Ca**]), the corresponding embedding problem is rather unexplored. A classical theorem asserts that every compact Riemann surface can be holomorphically embedded in $\mathbb{CP}^3$. The Minimal Set Problem, as Ghys has suggested [**Gh**], could be viewed as a special case of the embedding problem for RSL's: "Can a non-trivial Riemann surface lamination be holomorphically embedded in $\mathbb{CP}^2$?"

In what follows we show some basic properties of non-trivial minimal sets. This theory, due to Camacho, Lins Neto, and Sad [**CLS1**], was developed in part in the hope of arriving at a contradiction to the existence of non-trivial minimal sets.

4.2. Uniqueness of Minimal Sets. How many distinct non-trivial minimal sets, if any at all, can an SHFC on $\mathbb{CP}^2$ have?

THEOREM 4.2. *An SHFC on $\mathbb{CP}^2$ has at most one non-trivial minimal set.*

The proof of this nice fact is quite elementary, and is based on the study of the distance between two non-singular leaves and the application of the Maximum Principle for real harmonic functions (compare [**CLS1**]).

To study the distance between two leaves, we have to choose a suitable Riemannian metric on $\mathbb{CP}^2$. Consider the Hermitian metric

$$ds^2 = \frac{|dx|^2 + |dy|^2 + |xdy - ydx|^2}{(1 + |x|^2 + |y|^2)^2} \tag{4.1}$$

in the affine chart $(x, y) \in U_0$, which extends to a Hermitian metric on the entire projective plane. It is called the *Fubini-Study metric* on $\mathbb{CP}^2$ [**GH**]. We will denote by d the Riemannian distance induced by this metric. Note that the associated $(1, 1)$-form of the Fubini-Study metric is

$$\begin{aligned} \Omega &= \frac{\sqrt{-1}}{2\pi}\ \frac{dx \wedge d\overline{x} + dy \wedge d\overline{y} + (xdy - ydx) \wedge (\overline{x}d\overline{y} - \overline{y}d\overline{x})}{(1 + |x|^2 + |y|^2)^2} \\ &= \frac{\sqrt{-1}}{2\pi}\ \partial\overline{\partial} \log(1 + |x|^2 + |y|^2). \end{aligned} \tag{4.2}$$

LEMMA 4.3. *For any $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$ in the affine chart $(x, y) \in U_0$, we have*

$$d(p_0, p_1) \leq \frac{|p_0 - p_1|}{(1 + \delta^2(p_0, p_1))^{1/2}},$$

where $\delta(p_0, p_1)$ is the minimum (Euclidean) distance form the origin to the line segment which joins p_0 and p_1 .

PROOF. By definition, $d(p_0, p_1) = \inf_\gamma\{\int_0^1 \|\gamma'(t)\|dt\}$, where the infimum is taken over all piecewise smooth curves $\gamma : [0, 1] \to \mathbb{CP}^2$ with $\gamma(0) = p_0, \gamma(1) = p_1$. In particular, when $\gamma(t) = (1 - t)p_0 + tp_1 =: (x(t), y(t))$, one has

$$d^2(p_0, p_1) \leq (\int_0^1 \|\gamma'(t)\|dt)^2 \leq \int_0^1 \|\gamma'(t)\|^2 dt.$$

Now we estimate $\|\gamma'\|^2$:

$$\begin{aligned}
\|\gamma'(t)\|^2 &= \frac{|x'(t)|^2+|y'(t)|^2+|x(t)y'(t)-y(t)x'(t)|^2}{(1+|x(t)|^2+|y(t)|^2)^2} \\
&\leq \frac{|x'(t)|^2+|y'(t)|^2+(|x(t)|^2+|y(t)|^2)(|x'(t)|^2+|y'(t)|^2)}{(1+|x(t)|^2+|y(t)|^2)^2} \\
&= \frac{(|x'(t)|^2+|y'(t)|^2)(1+|x(t)|^2+|y(t)|^2)}{(1+|x(t)|^2+|y(t)|^2)^2} \\
&= \frac{|x_0-x_1|^2+|y_0-y_1|^2}{1+|x(t)|^2+|y(t)|^2} \\
&\leq \frac{|p_0-p_1|^2}{1+\delta^2(p_0,p_1)},
\end{aligned}$$

and this completes the proof. □

Corollary 4.4. *Let E and F be two disjoint compact subsets of $\mathbb{CP}^2$, and $E' := E\cap U_0$ and $F' := F\cap U_0$ be both non-empty. If $\epsilon := \inf\{|p-q| : (p,q)\in E'\times F'\}$, then $\epsilon>0$ and there exists a pair $(p,q)\in E'\times F'$ with $|p-q|=\epsilon$.*

Proof. Let $(p_n,q_n)\in E'\times F'$ be such that $|p_n-q_n|\to\epsilon$ as $n\to\infty$. By taking subsequences, if necessary, we may assume that $p_n\to p\in E$ and $q_n\to q\in F$. If $(p,q)\in E'\times F'$, we are done. Otherwise, if $p\in E\smallsetminus E'$, one has $q\in F\smallsetminus F'$ since $|p_n-q_n|$ is bounded. Therefore $\delta(p_n,q_n)\to\infty$ as $n\to\infty$, so $d(p,q)=0$ by Lemma 4.3, which is a contradiction. □

Proof of Theorem 4.2. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two non-trivial minimal sets of $\mathcal{F}$: $\{Pdy-Qdx=0\}$. They are necessarily disjoint by minimality. Set $\mathcal{M}_1' := \mathcal{M}_1\cap U_0$, $\mathcal{M}_2' := \mathcal{M}_2\cap U_0$ and $\epsilon := \inf\{|p-q| : (p,q)\in\mathcal{M}_1'\times\mathcal{M}_2'\}$. By Corollary 4.4, there exists $(p,q)\in\mathcal{M}_1'\times\mathcal{M}_2'$, with $|p-q|=\epsilon$. For simplicity, let $p=(0,0)$ and $q=(0,y_0)$, with $|y_0|=\epsilon$. It follows from the definition of ϵ that the y-axis is normal to $\mathcal{L}_p$ and $\mathcal{L}_q$ at p and q. Since p and q are not singular, we can parametrize the leaves by the x-parameter in a disk $\mathbb{D}(0,r)$ around the origin:

$$\mathcal{L}_p: \quad x\mapsto y_p(x), \quad y_p'(x)=\frac{Q(x,y_p(x))}{P(x,y_p(x))}, \quad y_p(0)=0,$$

$$\mathcal{L}_q: \quad x\mapsto y_q(x), \quad y_q'(x)=\frac{Q(x,y_q(x))}{P(x,y_q(x))}, \quad y_q(0)=y_0.$$

Define $h:\mathbb{D}(0,r)\to\mathbb{R}$ by $h(x)=\log|y_p(x)-y_q(x)|$. This is a harmonic function with a minimum at $x=0$. Therefore $h\equiv\log\epsilon$ on $\mathbb{D}(0,r)$, so that locally $\mathcal{L}_q$ is just the translation of $\mathcal{L}_p$ by y_0. By analytic continuation, this is true globally, i.e., $\mathcal{L}_q\cap U_0=(\mathcal{L}_p\cap U_0)+(0,y_0)$. By Corollary 2.16, there exists a sequence $q_n\in\mathcal{L}_q\cap U_0$ tending to infinity. The sequence $p_n := q_n-(0,y_0)\in\mathcal{L}_p\cap U_0$ also tends to infinity, so $d(p_n,q_n)\to 0$ by Lemma 4.3. This shows that $\overline{\mathcal{L}}_p\cap\overline{\mathcal{L}}_q=\mathcal{M}_1\cap\mathcal{M}_2\neq\emptyset$, which is a contradiction.

4.3. Poincaré Metric and Hyperbolicity. According to the Uniformization Theorem of Koebe-Poincaré-Riemann, every simply-connected Riemann surface is biholomorphic to one of the three standard models: The Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the unit disk $\mathbb{D}$ [**Ah**]. Since every Riemann surface has a holomorphic universal covering, it follows that *every Riemann surface can be covered holomorphically by $\widehat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$, in which case it is called spherical, Euclidean, or hyperbolic, respectively.* Elementary considerations show that every spherical Riemann surface is biholomorphic to $\widehat{\mathbb{C}}$, while the Euclidean Riemann surfaces are biholomorphic to $\mathbb{C}$, or to the cylinder $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$, or to a torus $\mathbb{C}/\Lambda$ for a rank 2 lattice Λ in the plane. The last two surfaces are biholomorphic to the quotients of $\mathbb{C}$ modulo the action of the groups generated by

$$\{z \mapsto z + a, \text{ for some } a \in \mathbb{C}^*\},$$

and

$$\{z \mapsto z + a, z \mapsto z + b, \text{ for some } a, b \in \mathbb{C}^* \text{ with } a/b \notin \mathbb{R}\},$$

respectively. All other Riemann surfaces are hyperbolic.

Definition 4.5. The Riemannian metric

$$\rho_{\mathbb{D}} := \frac{4}{(1-|z|^2)^2}\ |dz|^2$$

is called the *Poincaré* or *hyperbolic metric* on the unit disk.

The Poincaré metric $\rho_{\mathbb{D}}$ is invariant under all biholomorphisms $\varphi : \mathbb{D} \to \mathbb{D}$. It is uniquely characterized by this property up to multiplication by a non-zero constant. Therefore, all biholomorphisms of $\mathbb{D}$ are isometries with respect to the distance induced by $\rho_{\mathbb{D}}$. The unit disk equipped with this distance is a complete metric space [**M**].

Now let X be a hyperbolic Riemann surface, with the universal covering map $\pi : \mathbb{D} \to X$. Since the Poincaré metric $\rho_{\mathbb{D}}$ is invariant under all covering transformations, $\rho_{\mathbb{D}}$ induces a well-defined Poincaré metric ρ_X on X which is invariant under all biholomorphisms $X \to X$. By the definition of ρ_X, the projection π is a local isometry.

It is a direct consequence of Schwarz Lemma that if $\varphi : X \to Y$ is a holomorphic mapping between hyperbolic Riemann surfaces, then φ decreases the Poincaré distances, i.e., for every $x, y \in X$,

$$d_Y(\varphi(x), \varphi(y)) \le d_X(x, y),$$

where d_X and d_Y are the Riemannian distances induced by ρ_X and ρ_Y, respectively. If the equality holds for a pair (x, y), then φ will be a local isometry and a covering map [**M**].

Definition 4.6. For a conformal metric $ds^2 = h^2|dz|^2$ on a Riemann surface, the *Gaussian curvature* κ is given by

$$\kappa(z) = -\frac{(\Delta \log h)(z)}{h^2(z)},$$

where, as usual, z is a local uniformizing parameter on the surface.

It follows form this definition that the *Gaussian curvature is a conformal invariant*, that is, if $\varphi : X \to Y$ is a holomorphic map between Riemann surfaces, and if ds^2 is a conformal metric on Y, then at any point $z \in X$ for which $\varphi'(z) \neq 0$, the curvature κ' at z of the pull-back metric on X is equal to the curvature κ of ds^2 at $\varphi(z)$.

It also follows from Definition 4.5 and Definition 4.6 that the Poincaré metric $\rho_{\mathbb{D}}$ on the unit disk has constant Gaussian curvature -1. The same is true for every hyperbolic surface equipped with the Poincaré metric since curvature is a conformal invariant.

The fact that hyperbolic Riemann surfaces admit a metric of strictly negative curvature is a characteristic property, as seen in the following theorem:

THEOREM 4.7. *Suppose that X is a Riemann surface that has a conformal metric whose Gaussian curvature κ satisfies $\kappa < \sigma < 0$ for some constant σ. Then X is hyperbolic.*

Indeed, this is a special case of a more general fact: A complex manifold M which admits a distance for which every holomorphic mapping $\mathbb{D} \to M$ is distance-decreasing is hyperbolic in the sense of Kobayashi. For Riemann surfaces, the usual notion of hyperbolicity is equivalent to the hyperbolicity in the sense of Kobayashi. By Ahlfors' generalized version of Schwarz Lemma, a Riemann surface X which has a conformal metric of strictly negative curvature admits a distance for which every holomorphic mapping $\mathbb{D} \to X$ is distance-decreasing (see [**Kob**], and also [**Kr**] for an elegant exposition in the case of domains in $\mathbb{C}$).

4.4. Hyperbolicity in Minimal Sets. Our next goal is to determine the conformal type of a leaf in the non-trivial minimal set. To this end, let us construct a Hermitian metric on $\mathbb{CP}^2 \smallsetminus \text{sing}(\mathcal{F})$ which induces a conformal metric of negative Gaussian curvature on each non-singular leaf of a given SHFC $\mathcal{F}$. The metric is a modification of the Fubini-Study metric (4.1).

Suppose that $\mathcal{F} : \{\omega = Pdy - Qdx = 0\} \in \mathcal{D}_n$, and let $R = yP - xQ$. Consider the following Hermitian metric on $U_0 \smallsetminus \text{sing}(\mathcal{F})$:

$$\rho := (1 + |x|^2 + |y|^2)^{n-1} \frac{|dx|^2 + |dy|^2 + |xdy - ydx|^2}{|P(x,y)|^2 + |Q(x,y)|^2 + |R(x,y)|^2}. \tag{4.3}$$

ρ extends to a Hermitian metric on $\mathbb{CP}^2 \smallsetminus \text{sing}(\mathcal{F})$. To see this, let us for example compute the extended metric on the affine chart $(u,v) \in U_1$ (compare (1.6)):

$$\begin{aligned}(\phi_{10}^*\rho)(u,v) &= \left(1 + \frac{1}{|u|^2} + \frac{|v|^2}{|u|^2}\right)^{n-1} \frac{|u|^{-4}(|du|^2 + |udv - vdu|^2 + |dv|^2)}{|P(\frac{1}{u},\frac{v}{u})|^2 + |Q(\frac{1}{u},\frac{v}{u})|^2 + |R(\frac{1}{u},\frac{v}{u})|^2} \\ &= \left(1 + |u|^2 + |v|^2\right)^{n-1} \frac{|du|^2 + |dv|^2 + |udv - vdu|^2}{|\tilde{P}(u,v)|^2 + |\tilde{Q}(u,v)|^2 + |\tilde{R}(u,v)|^2},\end{aligned}$$

where $\tilde{P}(u,v) = u^{n+1}P\left(\frac{1}{u},\frac{v}{u}\right)$, $\tilde{Q}(u,v) = u^{n+1}Q\left(\frac{1}{u},\frac{v}{u}\right)$, and $\tilde{R}(u,v) = u^{n+1}R\left(\frac{1}{u},\frac{v}{u}\right)$ are polynomials in u, v.

Now let $p \in U_0$ be a non-singular point of $\mathcal{F}$, and let $\eta : T \mapsto (x(T), y(T))$ be a local parametrization of $\mathcal{L}_p$ near p with $\eta(0) = p$. By (4.3) above, the induced conformal metric on $\mathcal{L}_p$ is

$$ds^2 = (1+|x(T)|^2+|y(T)|^2)^{n-1} \frac{|x'(T)|^2 + |y'(T)|^2 + |x(T)y'(T) - y(T)x'(T)|^2}{|P(\eta(T))|^2 + |Q(\eta(T))|^2 + |R(\eta(T))|^2} |dT|^2$$

$$= (1 + |x(T)|^2 + |y(T)|^2)^{n-1}|dT|^2$$
$$=: h^2(T)|dT|^2.$$

By Definition 4.6 and conformal invariance of the curvature, the Gaussian curvature of $\mathcal{L}_p$ at p is given by

$$\kappa(p) = -\frac{(\Delta \log h)(0)}{h^2(0)}.$$

Computation gives

(4.4)

$$\begin{aligned}
\kappa(p) &= \frac{-2}{(1+|p|^2)^{n-1}} \left(\frac{\partial}{\partial T} \frac{\partial}{\partial \overline{T}} \log h^2(T) \right) \Big|_{T=0} \\
&= \frac{-2(n-1)}{(1+|p|^2)^{n-1}} \left(\frac{\partial}{\partial T} \frac{\partial}{\partial \overline{T}} \log(1 + |x(T)|^2 + |y(T)|^2) \right) \Big|_{T=0} \\
&= \frac{-2(n-1)}{(1+|p|^2)^{n-1}} \left(\frac{|x'(T)|^2 + |y'(T)|^2 + |x(T)y'(T) - y(T)x'(T)|^2}{(1 + |x(T)|^2 + |y(T)|^2)^2} \right) \Big|_{T=0} \\
&= \frac{-2(n-1)}{(1+|p|^2)^{n+1}} (|P(p)|^2 + |Q(p)|^2 + |R(p)|^2),
\end{aligned}$$

which is strictly negative.

Now let $\mathcal{F} \in \mathcal{D}_n$ have a non-trivial minimal set $\mathcal{M}$, and $p \in \mathcal{M}$. As the above expression is a continuous function of p which extends to $\mathbb{CP}^2 \smallsetminus \operatorname{sing}(\mathcal{F})$, it follows that the Gaussian curvature of the induced metric on $\mathcal{L}_p$ is uniformly bounded from above by a negative constant. By Theorem 4.7, we obtain

Theorem 4.8. *Every leaf in the non-trivial minimal set is a hyperbolic Riemann surface.*

Example 4.9. We can use the preceding result to show that no SHFC $\mathcal{F}$ of geometric degree 1 can have a non-trivial minimal set. Let $\mathcal{L}$ be a leaf of $\mathcal{F}$ contained in the non-trivial minimal set $\mathcal{M}$. Choose $p \in \mathcal{L}$ and let $T \mapsto \eta(T)$ be a local parametrization of $\mathcal{L}$ near p, with $\eta(0) = p$. The germ of η can be analytically continued over the entire plane $\mathbb{C}$. To see this, observe that $\mathcal{F}$ is induced by a holomorphic vector field on $\mathbb{CP}^2$ (see the comment after Theorem 1.18). By Proposition 1.9, this vector field lifts to a linear vector field on $\mathbb{C}^3$. Since every integral curve of a linear vector field is parametrized by $\mathbb{C}$, the same must be true for $\mathcal{L}$. It follows that the result of analytic continuation of η is a single-valued function. Now $\mathcal{L}$ is hyperbolic by Theorem 4.8. Let $\pi : \mathbb{D} \to \mathcal{L}$ be the universal covering map, and lift η to obtain a holomorphic map $\tilde{\eta} : \mathbb{C} \to \mathbb{D}$ with $\pi \circ \tilde{\eta} = \eta$. By Liouville's Theorem, $\tilde{\eta}$ is constant, which is a contradiction.

4.5. Algebraic Leaves and Minimal Sets. The next theorem answers a basic question which is of special interest in the case of any foliated manifold (compare [**CLS1**]).

Theorem 4.10. *No SHFC on $\mathbb{CP}^2$ can have a compact non-singular leaf.*

Proof. Let $\mathcal{L}$ be a compact non-singular leaf of $\mathcal{F} : \{Pdy - Qdx = 0\}$. By Chow's Theorem [**GH**], $\mathcal{L}$ is a smooth irreducible algebraic curve in $\mathbb{CP}^2$. Note that $\mathcal{L}$ cannot be a component of both curves $S_P : \{P = 0\}$ and $S_Q : \{Q = 0\}$, since P and Q are relatively prime. Hence, for example, we may assume that P is not identically zero on $\mathcal{L}$. It follows that the intersection of $\mathcal{L}$ with S_P is a finite set $\{p_1, \dots, p_k\}$.

Consider the 1-forms

$$\alpha = \frac{\partial}{\partial y}\left(\frac{Q}{P}\right) dx \qquad \text{and} \qquad \beta = -\partial \log(1 + |x|^2 + |y|^2)$$

in the affine chart $(x, y) \in U_0$. An easy computation shows that the 1-form $\tau = \alpha+\beta$ is well-defined on $\mathbb{CP}^2 \smallsetminus S_P$. For example, in the affine chart $(u, v) \in U_1$ it is given by

$$\frac{\partial}{\partial v}\left(\frac{\tilde{R}}{\tilde{P}}\right) du - \partial \log(1 + |u|^2 + |v|^2),$$

where $\tilde{P}$ and $\tilde{R}$ are polynomials in u, v defined in §4.4. The restriction $\alpha|_{\mathcal{L}}$ has poles at the finite set $\{p_1, \dots, p_k\}$ where $\mathcal{L}$ has a vertical tangent line. Without loss of generality we assume that all the p_j are in the affine chart U_0. Furthermore, it is easy to compute the residue of $\alpha|_{\mathcal{L}}$ at p_j: If $p_j = (x_j, y_j)$ and if $y \mapsto x_j + \sum_{i=m_j}^{\infty} a_i(y - y_j)^i$ is the local parametrization of $\mathcal{L}$ near p_j with $a_{m_j} \neq 0$, then

$$\mathrm{Res}[\alpha|_{\mathcal{L}}; p_j] = 1 - m_j.$$

Now consider small disks $D_j \subset \mathcal{L}$ around each p_j and integrate the 2-form $d\tau$ over $\mathcal{L}' = \mathcal{L} \smallsetminus \bigcup D_j$:

$$\begin{aligned} \int_{\mathcal{L}'} d\tau &= \sum \int_{\partial D_j} \tau \\ &= \sum \int_{\partial D_j} \alpha + \sum \int_{\partial D_j} \beta \\ &= (2\pi\sqrt{-1}) \sum (1 - m_j) + \sum \int_{\partial D_j} \beta. \end{aligned}$$

On the other hand, $d\tau|_{\mathcal{L}'} = d\alpha|_{\mathcal{L}'} + d\beta|_{\mathcal{L}'} = d\beta|_{\mathcal{L}'} = \bar{\partial}\beta|_{\mathcal{L}'} = (2\pi\sqrt{-1})\Omega|_{\mathcal{L}'}$, where Ω is the standard area form (4.2) coming from the Fubini-Study metric. Therefore

$$(2\pi\sqrt{-1})\ \mathrm{area}(\mathcal{L}') = (2\pi\sqrt{-1}) \sum (1 - m_j) + \sum \int_{\partial D_j} \beta.$$

Letting D_j shrink to p_j, we get

$$\mathrm{area}(\mathcal{L}) = \sum (1 - m_j) \leq 0,$$

which is a contradiction. $\square$

As a result, every algebraic leaf of an SHFC $\mathcal{F}$ must have some singular points of $\mathcal{F}$ in its closure. Note that every singularity of an algebraic leaf is indeed a singular point of $\mathcal{F}$ as well.

Proposition 4.11. *Let $\mathcal{M}$ be the non-trivial minimal set of an SHFC $\mathcal{F}$ on $\mathbb{CP}^2$. Then $\mathcal{M}$ intersects every algebraic curve in $\mathbb{CP}^2$.*

Proof. Let $S_K : \{K = 0\}$ be an algebraic curve in $\mathbb{CP}^2$ of degree k. For every triple (a, b, c) of positive real numbers, define

$$\varphi(x, y) = \varphi_{a,b,c}(x, y) := \frac{|K(x, y)|^2}{(a + b|x|^2 + c|y|^2)^k}$$

which is a non-negative real-analytic function on the affine chart $(x, y) \in U_0$. Since K has degree k, φ can be extended to a real analytic function on the entire $\mathbb{CP}^2$, with $S_K = \varphi^{-1}(0)$.

Suppose that $\mathcal{M} \cap S_K = \emptyset$. Then φ attains a positive minimum on $\mathcal{M}$, i.e., there exists $p_0 \in \mathcal{M}$ such that $\varphi(p) \geq \varphi(p_0) > 0$ for all $p \in \mathcal{M}$. Define $\psi : \mathcal{M} \to \mathbb{R}$ by $\psi(p) = \log \varphi(p)$. Clearly $\psi(p) \geq \log \varphi(p_0) > -\infty$ for all $p \in \mathcal{M}$. On the other hand, ψ is superharmonic along the non-singular leaf $\mathcal{L}_{p_0}$. To see this, let $\eta : T \mapsto (x(T), y(T))$ be a local parametrization of $\mathcal{L}_{p_0}$ near p_0, with $\eta(0) = p_0$. Then $\psi(\eta(T)) > -\infty$, and

$$\begin{aligned} \Delta\psi(\eta(T)) = \quad & 4\frac{\partial}{\partial T}\frac{\partial}{\partial \overline{T}}\psi(\eta(T)) \\ = \quad & -4k(a + b|x(T)|^2 + c|y(T)|^2)^{-2} \\ & (ab|x'(T)|^2 + ac|y'(T)|^2 + bc|x(T)y'(T) - y(T)x'(T)|^2) \end{aligned}$$

which is negative. Since $\psi(\eta(T))$ has a minimum at $T = 0$, it follows that φ is constant on $\mathcal{L}_{p_0}$, hence on $\mathcal{M}$ since $\overline{\mathcal{L}}_{p_0} = \mathcal{M}$. Therefore, for any triple (a, b, c) of positive real numbers, there exists $\alpha > 0$ such that

$$|K(x, y)|^{2/k} = \alpha(a + b|x|^2 + c|y|^2)$$

for all $(x, y) \in \mathcal{M}$. Evidently, this is impossible. □

Corollary 4.12. *No SHFC on $\mathbb{CP}^2$ which has an algebraic leaf can have a non-trivial minimal set.*

Proof. Let $\mathcal{L}$ be an algebraic leaf of an SHFC $\mathcal{F}$. By Theorem 4.10, $\overline{\mathcal{L}}$ necessarily contains a singular point of $\mathcal{F}$, say q. If $\mathcal{M}$ is a non-trivial minimal set of $\mathcal{F}$, then there exists $p \in \mathcal{M} \cap \mathcal{L}$ by Proposition 4.11. As p is non-singular, $\overline{\mathcal{L}} = \overline{\mathcal{L}}_p = \mathcal{M}$, so $q \in \mathcal{M} \cap \operatorname{sing}(\mathcal{F})$, which is a contradiction. □

Therefore, in order to find an $\mathcal{F}$ with a non-trivial minimal set, we must look for $\mathcal{F}$ in the sub-class of $\mathcal{D}_n$ consisting of SHFC's which do not admit any algebraic leaf. This sub-class is open and dense in $\mathcal{D}_n$ (see Remark 3.22).

Note that the above corollary gives another proof for the fact that no SHFC of geometric degree 1 can have a non-trivial minimal set, since one can easily see that every SHFC of geometric degree 1 has a projective line as a leaf.

We conclude with few important remarks.

Remark 4.13. It is shown in [**CLS1**] that every leaf $\mathcal{L}$ in the non-trivial minimal set $\mathcal{M}$ has *exponential growth*. This means that if we fix some Riemannian metric on $\mathcal{L}$ and some $p \in \mathcal{L}$, then

$$\liminf_{r \to +\infty} \frac{\log(\text{area}(B_r(p)))}{r} > 0,$$

where $B_r(p)$ denotes the open ball in $\mathcal{L}$ of radius r centered at p.

Remark 4.14. C. Bonatti, R. Langevin, and R. Moussu [**BLM**] have shown that for any non-trivial minimal set $\mathcal{M}$, there exists a leaf $\mathcal{L} \subset \mathcal{M}$ such that the monodromy group $G(\mathcal{L})$ contains a hyperbolic germ in $\text{Bih}_0(\mathbb{C})$. The real version of this theorem is a famous 1965 result of R. Sacksteder [**Sa**]: An exceptional minimal set of a transversely orientable codimension one C^2 foliation on a compact manifold contains a leaf with a hyperbolic monodromy mapping.

REMARK 4.15. Here is a related result due to A. Candel and X. Gómez-Mont [**CaG**] (see also the paper by A. Glutsyuk in [**I6**] for a generalization): Let $\mathcal{F}$ be an SHFC with hyperbolic singular points and no algebraic leaves. Then every leaf of $\mathcal{F}$ is a hyperbolic Riemann surface. In fact, a non-hyperbolic leaf gives rise to a non-trivial invariant transverse measure for $\mathcal{F}$. The support of this measure cannot intersect the leaves outside of the (possible) minimal set since then it has to be supported on the (global) separatrices by hyperbolicity of the singular points, and this means that these separatrices (with singular points added) are compact, hence algebraic, which is a contradiction. On the other hand, no invariant transverse measure can live on the minimal set by a result of [**CLS1**].

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