Comment on a Theorem of Kerékjártó

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"Every periodic homeomorphism of the open unit disk in the plane or the 2-sphere is topologically conjugate to a Euclidean isometry." **[C-K]**

The following argument gives a closely related result at the expense of an extra regularity assumption.

Theorem. *Let $K > 1$ and let G be a group of orientation-preserving K-quasiconformal homeomorphisms of a Riemann surface X. Then there exists a Riemann surface Y and a quasiconformal homeomorphism $h : X \to Y$ such that the corresponding action of G on Y (under conjugation by h) is given by conformal homeomorphisms.*

Proof. The existence of Y is equivalent to the existence of a conformal structure μ of bounded dilatation on X which is invariant under G. In fact, if $h \circ f \circ h^{-1} : Y \to Y$ is conformal for all f in G, set $\mu = h^*\sigma$, the pull-back of the standard conformal structure σ on Y. Conversely, given a μ such that $f^*\mu = \mu$ for all f in G, let $h : X \to Y$ be the solution of the Beltrami equation $h^*\sigma = \mu$, which exists by the measurable Riemann mapping theorem of Ahlfors and Bers **[A-B]** . Then $h \circ f \circ h^{-1}$ preserves σ for any f in G, hence is conformal.

To construct such an invariant μ, Let ν be any conformal structure of bounded dilatation on X, and define

$$\mathcal{C} = \{f^*\nu : f \in G\}.$$

On any local coordinate (U, z) on X, every element g of $\mathcal{C}$ is given by a Beltrami differential $g(z)d\bar{z}/dz$, where $g : U \to \mathbf{D}$ is a complex-valued measurable function, and

$$\sup_{g \in \mathcal{C}} \sup_{z \in U} |g(z)| < \epsilon < 1,$$

where ϵ depends only on K. Now fix a local coordinate (U, z) and define

$$E(z) = \text{hyperbolic convex hull of } \{g(z) : g \in \mathcal{C}\} \text{ in } \mathbf{D}.$$

Obviously $E(z)$ is a convex set compactly contained in $\mathbf{D}$. We will assign a barycenter to $E(z)$ following Douady and Earle. Let $m_{E(z)}$ be the Borel probability measure defined on the unit circle $\partial\mathbf{D}$ by

$$m_{E(z)}(A) = \frac{1}{\text{area}(E(z))} \int_{E(z)} \eta_w(A) d\rho(w) \qquad (A \subset \partial\mathbf{D} \text{ is a Borel set}),$$

where η_w is the harmonic measure on the unit circle associated with $w \in E(z)$ and $d\rho$ is the Poincaré metric on $\mathbf{D}$. (When $E(z)$ is just a geodesic, we modify the formula by using length instead of area.) Note that for every $\phi \in \mathrm{Aut}(\mathbf{D})$, $\phi_*(m_{E(z)}) = m_{\phi(E(z))}$ so the Möbius group $\mathrm{Aut}(\mathbf{D})$ respects the assignment $E(z) \mapsto m_{E(z)}$. According to Douady and Earle [**D-E**], to each Borel probability measure on the unit circle there corresponds a unique point in $\mathbf{D}$ called the *conformal barycenter* of the measure, and $\mathrm{Aut}(\mathbf{D})$ respects this assignment. Let us denote the conformal barycenter of $m_{E(z)}$ by $\mu_{E(z)}$. It follows that $|\mu_{E(z)}| < \epsilon$ and

$$\phi(\mu_{E(z)}) = \mu_{\phi(E(z))} \quad \text{for every } \phi \in \mathrm{Aut}(\mathbf{D}). \qquad (*)$$

Now define a conformal structure μ on X which is given by the Beltrami differential

$$\mu_{E(z)} \frac{d\bar{z}}{dz}$$

in any local coordinate (U, z). μ is well-defined since if ζ is another local coordinate on U, then $E(\zeta) = \phi(E(z))$, where $\phi \in \mathrm{Aut}(\mathbf{D})$ is the pure rotation $w \mapsto (\zeta_z/\bar{\zeta}_z)w$. So $\mu_{E(\zeta)}(d\bar{\zeta}/d\zeta) = \mu_{E(z)}(d\bar{z}/dz)$ by $(*)$. Obviously μ has bounded dilatation.

The invariance of μ under the action of G is now a simple consequence of $(*)$. Let $f \in G$ and write the local expression for $f^*\mu$:

$$(f^*\mu)(z) = \theta(z) \frac{\bar{\theta}(z)\mu_f(z) + \mu_{E(f(z))}}{1 + \theta(z)\bar{\mu}_f(z)\mu_{E(f(z))}} \left(\frac{d\bar{z}}{dz}\right),$$

where $\mu_f(z) = f_{\bar{z}}/f_z$ and $\theta(z) = \bar{f}_z/f_z$. Note that the mapping ϕ defined by

$$w \mapsto \theta(z) \frac{\bar{\theta}(z)\mu_f(z) + w}{1 + \theta(z)\bar{\mu}_f(z) w}$$

belongs to $\mathrm{Aut}(\mathbf{D})$. Therefore, by $(*)$,

$$(f^*\mu)(z) = \phi(\mu_{E(f(z))}) \frac{d\bar{z}}{dz} = \mu_{\phi(E(f(z)))} \frac{d\bar{z}}{dz} = \mu_{E(z)} \frac{d\bar{z}}{dz},$$

which proves $f^*\mu = \mu$. $\square$

Remarks.

(1) The Theorem proves Kerékjártó's result for a periodic quasiconformal homeomorphism, since there is a unique conformal structure on the disk and the 2-sphere and every periodic conformal homeomorphism of these two surfaces is conformally conjugate to an element of $SO(2)$ or $SO(3)$.

(2) It would be nice to know how far one can push this idea without the assumption of having a uniformly quasiconformal action. For the conformal structure μ to be integrable, it has to be only locally bounded. On the other hand, it is clear from the above argument that we can define μ even in the case where the convex set $E(z)$ is not compact, provided that $E(z)$ has interior and its hyperbolic area is finite. Simple examples show that this is necessary. For instance, let f be any diffeomorphism of

a Riemann surface which is indentity away from a small disk and acts contracting inside the disk with a fixed point at the center, and let G be the group generated by f. Then for z in the small disk, $E(z)$ is a geodesic of infinite length. Obviously there is no way to assign a barycenter to $E(z)$ and f cannot be conjugate to any conformal homeomorphism.

(3) Finally we ask if there is a similiar proof for Smale's theorem which asserts that $SO(3)$ is a deformation retract of the group $\text{Diff}^+(S^2)$.

References

[**A-B**] Ahlfors L. and Bers L., "Riemann's mapping theorem for variable metrics," *Ann. of Math.,* **72**(1960) 385-404.

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