Action of finite Blaschke products on the unit circle

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Consider a finite Blaschke product

$$f(z) = \lambda \prod_{j=1}^{d} \left(\frac{z - a_j}{1 - \overline{a_j} z} \right),$$

where $d \geq 2$, $|\lambda| = 1$ and $|a_j| < 1$. Then f is a degree d branched covering $\mathbb{D} \to \mathbb{D}$ so it induces a d-to-1 covering map of the unit circle $\mathbb{S}^1$.

Theorem. *Let f be a Blaschke product as above, with a (necessarily unique) fixed point $p \in \mathbb{D}$. Then the restriction $f|_{\mathbb{S}^1}$ has a unique absolutely continuous invariant probability measure given by*

$$\frac{1 - |p|^2}{|1 - \overline{p}\, e^{2\pi i t}|^2}\, dt.$$

(the harmonic measure on $\mathbb{S}^1$ as seen from p). In particular, if $f(0) = 0$, the unique absolutely continuous invariant measure for $f|_{\mathbb{S}^1}$ is Lebesgue measure dt.

Proof. It suffices to consider the case $p = 0$ and show that for every continuous function $\phi : \mathbb{S}^1 \to \mathbb{R}$

$$\int_0^1 (\phi \circ f)(e^{2\pi i t})\, dt = \int_0^1 \phi(e^{2\pi i t})\, dt.$$

Fix such a ϕ and let Φ be its harmonic extension to $\mathbb{D}$. The function $\Phi \circ f$ is also harmonic in $\mathbb{D}$ and extends $\phi \circ f$. Hence, by the mean-value property,

$$\int_0^1 (\phi \circ f)(e^{2\pi i t})\, dt = (\Phi \circ f)(0) = \Phi(0) = \int_0^1 \phi(e^{2\pi i t})\, dt.$$

This proves invariance of Lebesgue measure. To see uniqueness, it suffices to check that the action of $f|_{\mathbb{S}^1}$ is ergodic with respect to Lebesgue measure, for then any other invariant measure would be singular with respect to Lebesgue. To check ergodicity, suppose $E \subset \mathbb{S}^1$ is f-invariant, and consider the characteristic function $\phi = \chi_E$ which satisfies $\phi = \phi \circ f$ a.e. on $\mathbb{S}^1$. Let Φ be the Poisson integral of ϕ, which is bounded and harmonic in $\mathbb{D}$. The function $\Phi \circ f$ is also bounded and harmonic in $\mathbb{D}$, so it is the Poisson integral of some bounded measurable function g on $\mathbb{S}^1$. In particular, the radial limit of $\Phi \circ f$ is equal to g a.e. on $\mathbb{S}^1$. It easily follows that $g = \phi$ a.e. , hence $\Phi \circ f = \Phi$, hence $\Phi \circ f^{\circ n} = \Phi$ for all n. Since by Schwarz Lemma $f^{\circ n}(z) \to 0$ for every $z \in \mathbb{D}$, we obtain $\Phi(z) = \lim \Phi \circ f^{\circ n}(z) = \Phi(0)$, implying that Φ is constant in $\mathbb{D}$. Hence ϕ must be constant a.e. on $\mathbb{S}^1$. $\square$

Here are two more proofs for the fact that Lebesgue measure on $\mathbb{S}^1$ is f-invariant when $f(0) = 0$.

Proof 2. We show that for every measurable set $E \subset \mathbb{S}^1$,

$$\int_{|z|=1} (\chi_E \circ f)(z) \, \frac{dz}{iz} = \int_{|z|=1} \chi_E(z) \, \frac{dz}{iz}, \tag{1}$$

where χ_E is the characteristic function of E. Expand χ_E into its Fourier series

$$\chi_E(z) = \sum_{n=-\infty}^{\infty} c_n \, z^n, \tag{2}$$

where $2\pi c_0 = |E|$. We prove that for every integer $n \neq 0$

$$\int_{|z|=1} f(z)^n \, \frac{dz}{z} = 0. \tag{3}$$

Then (1) follows by substituting (2) for χ_E and integrating term by term. To prove (3), first suppose n is positive. Then $f(z)^n/z$ is holomorphic in a neighborhood of the closed unit disk since by $f(0) = 0$ the singularity at 0 is removable. In this case, (3) follows immediately from Cauchy's theorem. Now suppose n is negative. Then

$$\overline{\int_{|z|=1} f(z)^n \, \frac{dz}{z}} = \int_{|z|=1} z \, \overline{f(z)}^n \, d\overline{z} = \int_{|z|=1} z \, f(z)^{-n} \, d\left(\frac{1}{z}\right) = -\int_{|z|=1} f(z)^{-n} \, \frac{dz}{z}$$

and the last integral is zero by the first case above since $-n$ is positive.

Proof 3. A measure $\rho(t)\,dt$ with continuous density $\rho(t)$ is invariant under a local diffeomorphism $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ iff for every x,

$$\rho(x) = \sum_{g(t)=x} \frac{\rho(t)}{|g'(t)|}.$$

If $g(t) = \frac{1}{2\pi i} \log f(e^{2\pi i t})$, we have

$$|g'(t)| = g'(t) = \frac{1}{2\pi i} \, \frac{d \log f(e^{2\pi i t})}{dt} = \frac{e^{2\pi i t} f'(e^{2\pi i t})}{f(e^{2\pi i t})}.$$

It follows that Lebesgue measure dt is invariant under $f|_{\mathbb{S}^1}$ iff for every $\zeta \in \mathbb{S}^1$,

$$1 = \sum_{f(z_i)=\zeta} \frac{\zeta}{z_i \, f'(z_i)}. \tag{4}$$

To see (4), consider the meromorphic 1-form

$$\omega = \frac{dz}{z(f(z) - \zeta)}$$

on the sphere. Note that ω is holomorphic in a neighborhood of ∞. By the residue theorem,

$$\begin{aligned} 0 &= \mathrm{Res}[\omega; 0] + \sum_i \mathrm{Res}[\omega; z_i] \\ &= -\frac{1}{\zeta} + \sum_i \frac{1}{z_i f'(z_i)} \end{aligned}$$

which is equivalent to (4).

Remark 1. The existence of *some* absolutely continuous invariant probability measure for $f|_{\mathbb{S}^1}$ can be proved in a different way: Assuming $f(0) = 0$, we can write

$$f(z) = \lambda\, z^m \prod_{j=1}^{d-m} \left(\frac{z - a_j}{1 - \overline{a_j} z} \right),$$

where $1 \leq m \leq d$ and $0 < |a_j| < 1$. Then $f|_{\mathbb{S}^1}$ is expanding since a brief computation shows that when $|z| = 1$,

$$|f'(z)| = \frac{z f'(z)}{f(z)} = m + \sum_{j=1}^{d-m} \frac{1 - |a_j|^2}{|z - a_j|^2},$$

and this is uniformly greater than 1 on $\mathbb{S}^1$. It is well-known that a smooth expanding map of the circle admits an absolutely continuous invariant probability measure.

Remark 2. When the Blaschke product f has no fixed point in $\mathbb{D}$, I suspect that there is no absolutely continuous invariant probability measure for $f|_{\mathbb{S}^1}$. In this case, f has a non-repelling fixed point $p \in \mathbb{S}^1$ and we have two cases:

1. The Julia set $J(f)$ is a Cantor set of measure zero in $\mathbb{S}^1$, the Fatou set consists of a single component which is the basin of attraction of the attracting or parabolic fixed point p. In this case, it is easy to check that every f-invariant probability measure on $\mathbb{S}^1$ is supported on $J(f)$, hence it must be singular.

2. $J(f) = \mathbb{S}^1$, the Fatou set has two components $\mathbb{D}$ and $\widehat{\mathbb{C}} \smallsetminus \mathbb{D}$, each forming an attracting petal for the parabolic fixed point p. It is not hard to see that in this case there can be no f-invariant probability measure $\rho(t)\, dt$ on $\mathbb{S}^1$ with *continuous* density ρ. This continuity assumption is perhaps redundant, as the argument is likely to extend to $\rho \in L^1$.