

ANALYSIS ARCHIVE

Preface.

The present manuscript consists of 125 problems in Real and Complex Analysis, as well as hints and complete solutions, which is the outcome of a short problem-solving seminar that I ran during May-June 1999 at Stony Brook. The goal of this seminar was to prepare first-year graduate students for their written comprehensive exam. The problems are gathered from all sorts of references and have been modified, when necessary, to be made compatible with the scope of the comprehensive exam at Stony Brook. I am grateful to the participants in the seminar, especially to J. Friedman for his criticisms and comments on various drafts of the manuscript. I hope this collection will grow gradually and turn into a useful reference in the future.

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List of notations

$\#A$	the number of elements in a finite set A
\mathbb{D}	open unit disk $\{z \in \mathbb{C} : z < 1\}$
$\mathbb{D}(a, r)$	open disk $\{z \in \mathbb{C} : z - a < r\}$
\mathbb{H}	the upper half-plane $\{z \in \mathbb{C} : \text{Im}(z) > 0\}$
m	Lebesgue measure on \mathbb{R}^n
T^*	adjoint of a linear transformation T
\mathbb{T}	the circle $\mathbb{R}/(2\pi\mathbb{Z})$

I. Problems

A. Basic Real Analysis

A1. Characterize all functions from \mathbb{R} to \mathbb{R} which can be uniformly approximated on the real line by a sequence of polynomials.

A2. Let $f, g : [0, 1] \rightarrow [0, 1]$ be two continuous functions. If g is non-decreasing and $f \circ g = g \circ f$, prove that f and g have a common fixed point in $[0, 1]$.

A3. Let (X, d) be a compact metric space. Let $f : X \rightarrow X$ be a map that satisfies $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. Prove that f is an isometry $X \xrightarrow{\cong} X$.

A4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be one-to-one. Show that there exists a $c \in \mathbb{R}$ such that $f(c^2) - (f(c))^2 < 1/4$.

A5. Prove or disprove: There exists a non-empty perfect (=closed with no isolated point) subset of the unit circle which contains no rational angle.

A6. For $0 < x < 1$, let $0.x_1x_2x_3\cdots$ be the unique non-terminating decimal expansion of x (for example $1/5 = 0.1999\cdots$). What is the probability that for a randomly chosen x , the digit 2 appears before the digit 3 in the expansion $0.x_1x_2x_3\cdots$?

A7. Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous and $|f(x)| \leq \int_0^x f(t) dt$ for $0 \leq x \leq 1$. Show that $f \equiv 0$.

A8. Let I be the set of positive integers that do not contain the digit 9 in their decimal expansions. Prove that $\sum_{n \in I} 1/n$ is convergent.

A9. Let P be a non-constant polynomial with real coefficients and only real roots. Prove that for each $r \in \mathbb{R}$, the polynomial $Q_r(x) := P(x) - rP'(x)$ has only real roots.

A10. Find the largest term in the sequence $\{x_n\}$ defined by $x_n := \frac{1000^n}{n!}$ for $n = 1, 2, 3, \dots$.

A11. Let p be a positive integer. Determine the real numbers r for which every sequence $\{x_n\}_{n=0}^\infty$ of real numbers satisfying the relation $\frac{1}{2}(x_{n+1} + x_{n-1}) = rx_n$ has period p , i.e., $x_n = x_{n+p}$ for all $n \geq 0$.

A12. Let $\{a_n\}_{n \geq 1}$ be a sequence of complex numbers with the property that for

every sequence $\{b_n\}_{n \geq 1}$ with $\sum_{n=1}^{\infty} |b_n|^2 < \infty$ one has $\sum_{n=1}^{\infty} |a_n b_n| < \infty$. Prove that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.

A13. Can the interval $[0, 1] \subset \mathbb{R}$ be written as a union of countably infinite number of disjoint closed sets?

A14. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function which has a local minimum at every point. Prove that f must be constant.

A15. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $\lim_{n \rightarrow +\infty} f(nx) = 0$ for every $1 \leq x \leq 2$. Show that $\lim_{x \rightarrow +\infty} f(x) = 0$.

A16. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a positive continuous function. Show that

$$I := \left(\int_0^1 f(x) dx \right) \left(\int_0^1 \frac{1}{f(x)} dx \right) \geq 1.$$

B. Measures and Integrals

B1. Let E_1, E_2, \dots, E_n be measurable subsets of $[0, 1]$ such that every point in $[0, 1]$ belongs to at least q of these sets. Show that at least one E_i has measure $\geq q/n$.

B2. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a measurable function. Suppose that for every interval $J \subset [0, 1]$ we have $0 \leq \int_J f(x) dx \leq m(J)$. Prove that $0 \leq f \leq 1$ almost everywhere.

B3. Let $r_k = p_k/q_k$ be an enumeration of the rational numbers (in reduced form) in the interval $[0, 1]$. Define the functions $f_k(x) := \exp\{-(p_k - q_k x)^2\}$. Prove that $f_k \rightarrow 0$ in measure on $[0, 1]$, but $\lim_{k \rightarrow \infty} f_k(x)$ does not exist at any point $x \in [0, 1]$.

B4. Define a function f on $[0, 1]$ as follows: If $x = 0.x_1 x_2 x_3 \dots$ is the unique non-terminating decimal expansion of x , then $f(x) := \max_n x_n$. Show that f is measurable and find a simple description for it.

B5. Find the Lebesgue measure of the subset E of the square $[-1, 1] \times [-1, 1]$ in the plane consisting of points (x, y) such that $|\sin x| < 1/2$ and $\cos(x+y)$ is irrational.

B6. Suppose that $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are given functions, with g measurable and f continuous. Should the composition $g \circ f$ be measurable?

B7. Let $f : [0, 1] \rightarrow \mathbb{R}$ be continuous, and for every $y \in \mathbb{R}$ let $0 \leq n(y) \leq +\infty$

be the number of solutions to $f(x) = y$. Prove that $y \mapsto n(y)$ is a measurable function.

B8. Let $\nu \geq 2$ and define “Diophantine numbers of order ν ” as the set D_ν of all irrational numbers $0 < x < 1$ for which there exists a constant $C = C(x) > 0$ such that $|x - p/q| \geq C/q^\nu$ for all rationals $p/q \in [0, 1]$. Prove that when $\nu > 2$, almost every number in $[0, 1]$ belongs to D_ν . (On the other hand, D_2 has measure zero. This statement is harder to prove, but it is worth trying.)

B9. Let E_n be the set of real numbers $0 < x < 1$ whose unique non-terminating decimal expansion contains a string of zeros of length $\geq \log n$ starting at the n -th digit. What is the measure of the set E of points which belong to infinitely many of the set E_n ?

B10. Let $\mathbb{Q}^n \subset \mathbb{R}^n$ be the set of points with rational coordinates. If $E \subset \mathbb{R}^n$ has positive (n -dimensional Lebesgue) measure, what can be said about the measure of the set $F := \mathbb{R}^n \setminus \bigcup_{x \in \mathbb{Q}^n} (x + E)$?

B11. Let $K \subset \mathbb{R}^2$ be compact and convex, and let $K_\varepsilon := \bigcup_{x \in K} \mathbb{D}(x, \varepsilon)$ be its ε -neighborhood. Prove that the area of K_ε is of the form $a + b\varepsilon + c\varepsilon^2$ for some constants a, b, c . Can you identify these constants?

B12. Let (X, μ) be a measure space with $\mu(X) < \infty$. Let \mathcal{M} be the space of all complex-valued measurable functions on X (where, as usual, two functions are identified if they coincide except on a set of measure zero). For $f \in \mathcal{M}$ define

$$\rho(f) := \int_X \frac{|f|}{|f| + 1} d\mu.$$

Show that $(f, g) \mapsto \rho(f - g)$ defines a metric on \mathcal{M} . Show that a sequence f_n converges to f in this metric if and only if f_n converges to f in measure.

B13. Define the *Gauss map* $G : (0, 1] \rightarrow [0, 1)$ by $G(x) := 1/x - [1/x]$ (also known as the *fractional part* of $1/x$). Consider the probability measure μ on $[0, 1]$ defined by

$$d\mu := \frac{1}{\log 2} \left(\frac{1}{1+x} \right) dx.$$

Show that μ is an invariant measure for G , i.e., $\mu(E) = \mu(G^{-1}(E))$ for every measurable set $E \subset [0, 1]$.

B14. Recall that a metric space is *separable* if it contains a countable dense subset. Prove or disprove: $L^\infty[0, 1]$ is separable.

B15. Let (X, μ) be a measure space with $\mu(X) < \infty$. Consider an $f \in L^\infty(X, \mu)$ with $\|f\|_\infty \neq 0$. Define $p_n := \int_X |f|^n d\mu$. Prove that

$$\lim_{n \rightarrow \infty} \sqrt[n]{p_n} = \lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = \|f\|_\infty.$$

B16. Let $f \in L^p(\mathbb{R})$ for some $p \geq 1$. Define $f_\varepsilon(x) := f(x + \varepsilon)$. Prove that

$$\lim_{\varepsilon \rightarrow 0} \|f - f_\varepsilon\|_p = 0.$$

B17. Recall that a function $f : [0, 1] \rightarrow \mathbb{R}$ is M -Lipschitz if $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [0, 1]$. Prove that f is M -Lipschitz if and only if there exists a sequence $\{f_n\}$ of continuously differentiable functions defined on $[0, 1]$ such that

- $|f'_n(x)| \leq M$ for all n and all $x \in [0, 1]$, and
- $f_n(x) \rightarrow f(x)$ for all $x \in [0, 1]$ as $n \rightarrow \infty$.

B18. Let (X, μ) be a measure space with $\mu(X) < \infty$, and let $f_n \in L^1(X, \mu)$ converge to a measurable function f at almost every $x \in X$. Assume that there exist constants $C > 0$ and $p > 1$ such that

$$\sup_{n \geq 1} \int_X |f_n|^p d\mu < C.$$

Prove that $f \in L^1(X, \mu)$ and $\|f_n - f\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

B19. Let f be a bounded measurable function on a measure space (X, μ) . Assume that there exist constants $C > 0$ and $0 < \alpha < 1$ such that

$$\mu\{x \in X : |f(x)| > \varepsilon\} < \frac{C}{\varepsilon^\alpha}$$

for every $\varepsilon > 0$. Show that $f \in L^1(X, \mu)$.

B20. Let (X, μ) be a measure space with $\mu(X) < \infty$ and $\{f_n\}$ be a sequence of real-valued measurable functions on X . Let $\{\alpha_n\}$ be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \mu(\{x \in X : |f_n(x)| > \alpha_n\}) < \infty.$$

Prove that

$$-1 \leq \liminf_{n \rightarrow \infty} \frac{f_n(x)}{\alpha_n} \leq \limsup_{n \rightarrow \infty} \frac{f_n(x)}{\alpha_n} \leq 1$$

for μ -almost every $x \in X$.

B21. Let $E \subset \mathbb{R}^n$ be a set of positive (n -dimensional Lebesgue) measure. Prove that the arithmetical difference set

$$E - E := \{z \in \mathbb{R}^n : z = x - y \text{ for some } x, y \in E\}$$

contains an open ball (which may be chosen around the origin).

B22. Let (X, μ) be a measure space and $f : X \rightarrow \mathbb{R}$ be measurable.

(a) If $\mu(X) < \infty$, show that $f \in L^1(X, \mu)$ if and only if

$$\sum_{n=1}^{\infty} 2^n \mu\{x \in X : |f(x)| \geq 2^n\} < \infty.$$

(b) If $\mu(X) = \infty$ but f is bounded, show that $f \in L^1(X, \mu)$ if and only if

$$\sum_{n=1}^{\infty} 2^{-n} \mu\{x \in X : |f(x)| \geq 2^{-n}\} < \infty.$$

B23. Let $A = [a_{ij}]$ be a real symmetric $n \times n$ matrix, and define $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$f(x_1, x_2, \dots, x_n) := \exp\left\{-\sum_{i,j} a_{ij} x_i x_j\right\}.$$

Prove that $f \in L^1(\mathbb{R}^n)$ if and only if A is positive-definite, in which case $\|f\|_1 = \pi^{n/2}/\sqrt{\det A}$.

B24. Let $f_n \in L^2[0, 1]$ for $n = 1, 2, 3, \dots$ and suppose that $A \leq \|f_n\|_1 \leq \|f_n\|_2 \leq B$ for some constants $A, B > 0$ and all n . If the series $\sum_{n=1}^{\infty} c_n f_n$ converges absolutely for almost every $x \in [0, 1]$, show that $\sum_{n=1}^{\infty} |c_n|$ converges.

B25. Let $C[0, 1]$ denote the space of continuous real-valued functions on $[0, 1]$. Let S be the set of all $g \in C[0, 1]$ such that $g \equiv 0$ on $[0, \delta]$ for some $\delta > 0$ (depending on g). Suppose that $f_n \in C[0, 1]$ is a sequence of functions such that $f_n \geq 0$ and

$$\Lambda g := \lim_{n \rightarrow \infty} \int_0^1 g f_n$$

exists for all $g \in C[0, 1]$. Assuming that $\Lambda g = 0$ for all $g \in S$, prove that there exists a constant $M \geq 0$ such that $\Lambda g = M g(0)$ for all $g \in C[0, 1]$.

B26. Let $\Omega \subset \mathbb{R}^n$ be open, $\varphi : \Omega \rightarrow \mathbb{R}$ be measurable, and $1 \leq p \leq +\infty$. Suppose that for every $f \in L^p(\Omega)$, $\varphi f \in L^p(\Omega)$. Show that $\varphi \in L^\infty(\Omega)$.

B27. Let X be a complete metric space and $f_n : X \rightarrow \mathbb{C}$ be a sequence of continuous functions which at every point $x \in X$ converge to a well-defined limit $f(x) \in \mathbb{C}$.

(a) Show that there exists a non-empty open set $U \subset X$ and a constant $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in U$ and all $n \geq 1$.

(b) Show that for every $\varepsilon > 0$ there exists a non-empty open set $U \subset X$ and an integer N such that $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in U$ and all $n \geq N$.

C. Banach and Hilbert Spaces

C1. Let H be a complex Hilbert space and $f, g : H \rightarrow \mathbb{C}$ be bounded linear functionals with $\|f\| = \|g\|$. Suppose that $f(x) = 0$ implies $g(x) = 0$. Show that $f = e^{ir}g$ for some real number r .

C2. Let K be a closed, bounded, convex set in a Banach space and let $T : K \rightarrow K$ be a map which satisfies $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. Prove that for every $\varepsilon > 0$ there exists a point x in K such that $\|Tx - x\| < \varepsilon$. Does T necessarily have a fixed point?

C3. Let E be a normed linear space and $f : E \rightarrow \mathbb{C}$ be a bounded linear functional. Prove that $\|f\|$ is the reciprocal of the distance from the origin to the hyperplane $f(x) = 1$.

C4. Let E be a normed linear space and $F \subset E$ be a closed subspace. On the quotient E/F define the norm $\|x + F\| := \inf_{y \in x+F} \|y\|$. Prove or disprove: If F and E/F are Banach spaces, then so is E .

C5. Let H be a Hilbert space and $T : H \rightarrow H$ be a self-adjoint linear operator, i.e., $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$. Prove that T is continuous. Does the conclusion hold if H is only an inner product space?

C6. Let E be an inner product space (not necessarily Hilbert) and $F \subset E$ be a proper closed subspace. Does there exist a non-zero vector in E which is orthogonal to F ?

C7. Let H be a complex Hilbert space and $T : H \rightarrow H$ be a bounded linear operator such that $\lim_{\|x\| \rightarrow \infty} |\langle Tx, x \rangle| / \|x\| = +\infty$. Prove that T is an isomorphism.

C8. Let E be a normed linear space and $T : E \rightarrow E$ be a linear invertible operator with the property that there exists a $C > 0$ such that $\|T^n x\| \leq C\|x\|$ for all $x \in E$ and $n \in \mathbb{Z}$. Prove that there exists a norm $\|\cdot\|_T$ on E , equivalent to $\|\cdot\|$, with respect to which T is an isometry.

C9. (a) Consider the polynomials

$$T_n(x) = \frac{1}{2^{n-1}} \cos(n \cos^{-1} x)$$

for $n = 1, 2, 3, \dots$. Verify that T_n is monic, $\sup_{-1 \leq x \leq 1} |T_n(x)| = 1/2^{n-1}$, and T_n takes the values $\pm 1/2^{n-1}$ alternately at the points $\cos(k\pi/n)$, $k = 0, 1, \dots, n$.

(b) Let $C[-1, 1]$ be the Banach space of continuous real-valued functions on $[-1, 1]$ equipped with the sup norm. Let $P_n \subset C[-1, 1]$ be the subspace of polynomials of degree $< n$. Find the distance from the point $x^n \in C[-1, 1]$ to P_n .

C10. Let $\{A_n\}$ be a sequence of linear operators on a Hilbert space H which converges weakly to a linear operator A (i.e., for every $x \in H$ and every bounded linear functional $f : H \rightarrow \mathbb{C}$, $f(A_n x) \rightarrow f(Ax)$). Assume further that $\|A_n x\| \rightarrow \|Ax\|$ for all $x \in H$. Prove that A_n converges to A strongly (i.e., $A_n x \rightarrow Ax$ for every $x \in H$). Conclude that the notions of weak and strong convergence of a sequence of unitary operators to a unitary operator are equivalent.

C11. Does there exist a norm $\|\cdot\|$ on the space $C[0, 1]$ of continuous functions on $[0, 1]$ with respect to which $\|f_n - f\| \rightarrow 0$ if and only if $f_n(x) \rightarrow f(x)$ for all $x \in [0, 1]$?

D. Fourier Series and Integrals

D1. Find all continuously differentiable functions $f : \mathbb{T} \rightarrow \mathbb{R}$ such that

$$\int_0^{2\pi} f(x) dx = \int_0^{2\pi} (f(x) + f'(x))(f(x) - f'(x)) dx = 0.$$

D2. Let $f \in L^1(\mathbb{R})$ and define its *Fourier transform* by

$$\hat{f}(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixs} dx$$

for $s \in \mathbb{R}$. Suppose that $f \geq 0$ almost everywhere and $|\hat{f}(s)| \geq \hat{f}(0)$ for *some* $s \neq 0$. Show that $f = 0$ almost everywhere.

D3. Let γ be a smooth Jordan curve in the plane parametrized by the smooth function $f : \mathbb{T} \rightarrow \mathbb{R}^2$. If A denotes the area of the region bounded by γ and L denotes the length of γ , use the Fourier series of f to prove the *Isoperimetric Inequality*: $L^2 \geq 4\pi A$.

D4. Let $0 < \theta < 1$ be irrational and suppose that there is a measurable set $E \subset \mathbb{T}$ which is invariant under the rigid rotation $R_\theta(x) = x + 2\pi\theta \pmod{2\pi}$, i.e., $x \in E$ if and only if $R_\theta(x) \in E$. Prove that either E or its complement has measure zero on the circle. Is this statement true when θ is rational?

D5. Let $f : \mathbb{T} \rightarrow \mathbb{C}$ be a function of Hölder class $0 < \alpha \leq 1$, i.e., $|f(x) - f(y)| \leq M|x - y|^\alpha$ for some $M > 0$ and all x, y . If $\hat{f}(n)$ denotes the n -th Fourier coefficient of f , show that $|\hat{f}(n)| \leq (\text{const.})/|n|^\alpha$ for all $n \in \mathbb{Z}$.

D6. Let $f : \mathbb{T} \rightarrow \mathbb{C}$ be a function of bounded variation. If $\hat{f}(n)$ denotes the n -th Fourier coefficient of f , show that $|n\hat{f}(n)| \rightarrow 0$ as $|n| \rightarrow \infty$.

D7. Let H be the space of all functions $f \in L^2(\mathbb{T})$ such that

$$\|f\|_H := \left(\sum_{n=-\infty}^{\infty} (1+n^2)|\hat{f}(n)|^2 \right)^{\frac{1}{2}} < \infty,$$

where the $\hat{f}(n)$ denote the Fourier coefficients of f . Prove that there exists a universal constant $C > 0$ such that

$$\|f\|_{\infty} \leq C\|f\|_H$$

for all $f \in H$.

D8. Suppose that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^2 smooth, doubly periodic (i.e., $f(x+1, y) = f(x, y) = f(x, y+1)$ for all $x, y \in \mathbb{R}^2$), and not identically zero. If f is an eigenfunction for the Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, show that the corresponding eigenvalue is non-positive.

D9. Let $f \in L^2(\mathbb{T})$ and for $\varepsilon \in \mathbb{R}$ define $f_{\varepsilon}(x) := f(x + \varepsilon)$. Show that

$$\liminf_{\varepsilon \rightarrow 0} \frac{\|f - f_{\varepsilon}\|_2}{|\varepsilon|} > 0$$

unless f is constant almost everywhere.

D10. Let $C(\mathbb{T})$ denote the Banach space of continuous complex-valued functions on the unit circle \mathbb{T} (equipped with the sup norm) and $I \subset \mathbb{Z}$ be any index set. Let $S \subset C(\mathbb{T})$ be the linear span of $\{e^{ikx} : k \in I\}$ and

$$V := \{f \in C(\mathbb{T}) : \int_0^{2\pi} f(x)e^{-ikx} dx = 0 \text{ for all } k \notin I\}.$$

Show that S is dense in V .

D11. Let $f : \mathbb{T} \rightarrow \mathbb{C}$ be a continuous function defined on the unit circle with Fourier coefficients $\hat{f}(n)$. Show that for every $\varepsilon > 0$ there exists a trigonometric polynomial $P = \sum_{n=-N}^N \hat{P}(n)e^{inx}$ such that $\|P - f\|_{\infty} < \varepsilon$ and $|\hat{P}(n)| \leq |\hat{f}(n)|$ for all n .

D12. Let $f : \mathbb{T} \rightarrow \mathbb{C}$ be absolutely continuous with $f' \in L^2(\mathbb{T})$. Show that

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq \|f\|_1 + \frac{\pi}{\sqrt{3}} \|f'\|_2.$$

E. Basic Complex Analysis

E1. Prove that all zeros of the polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ lie in the open disk $\mathbb{D}(0, r)$, where $r := \sqrt{1 + |a_{n-1}|^2 + \cdots + |a_1|^2 + |a_0|^2}$.

E2. Let z_1, z_2, \dots, z_n be given complex numbers. Prove that there exists a set $I \subset \{1, 2, \dots, n\}$ such that

$$\left| \sum_{k \in I} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^n |z_k|.$$

E3. Let P be a monic polynomial with real coefficients and $P(0) = -1$. Assume that P has no roots in the open unit disk \mathbb{D} . Find $P(1)$.

E4. Let z_1, z_2, \dots, z_n be given complex numbers. Prove that there exists a real number $0 < t < 1$ such that

$$\left| 1 - \sum_{k=1}^n z_k e^{2\pi i k t} \right| \geq 1.$$

E5. Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map. Suppose that for some $a \in \mathbb{C}$, $f^{-1}(a) = \{a\}$. What can you say about the rational map

$$g(z) := \frac{1}{f\left(\frac{1}{z} + a\right) - a}?$$

E6. Let $P(z) = (z - a_1)(z - a_2) \cdots (z - a_n)$, where $n \geq 2$ and the a_j are distinct complex numbers. Prove that

$$\sum_{j=1}^n \frac{1}{P'(a_j)} = 0.$$

E7. Let $P : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial of degree $n \geq 1$ whose roots belong to the open unit disk. Let $P^*(z) := z^n \overline{P(1/\bar{z})}$. Prove that all roots of the polynomial $z \mapsto P(z) + P^*(z)$ belong to the unit circle.

F. Properties of Holomorphic and Harmonic Functions

F1. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be holomorphic and assume that $\|f\|_2 := \left(\int_{\mathbb{D}} |f(z)|^2 dx dy\right)^{1/2} < \infty$. Show that for all $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{1}{\sqrt{\pi}(1 - |z|)} \|f\|_2.$$

F2. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be holomorphic and suppose there is a constant $M > 0$ such that

$$\int_0^{2\pi} |f'(re^{i\theta})| d\theta \leq M$$

for all $r < 1$. Prove that

$$\int_{[0,1]} |f(x)| dx < \infty.$$

F3. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic. Assume that f is real on the real axis and has positive imaginary part on the upper half-plane. Prove that $f'(x) > 0$ for all real x .

F4. Prove that for every $\zeta \in \mathbb{C}$,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{2\zeta \cos \theta} d\theta = \sum_{k=0}^{\infty} \left(\frac{\zeta^k}{k!} \right)^2.$$

F5. Let $u : \mathbb{C} \rightarrow \mathbb{R}$ be a harmonic function such that for all $z \in \mathbb{C}$,

$$u(z) \leq a |\log |z|| + b,$$

where $a, b > 0$ are constants. Prove that u must be constant.

F6. Let f and g be holomorphic functions defined on an open neighborhood of the origin 0 in \mathbb{C} . Suppose that the function $h := f + \bar{g}$ fails to be one-to-one in any neighborhood of 0 . Show that $|f'(0)| = |g'(0)|$.

F7. Let (X, μ) be a measure space with $\mu(X) < \infty$ and U be a domain in \mathbb{C} . Let $f : X \times U \rightarrow \mathbb{C}$ be a bounded function such that $t \mapsto f(t, z)$ is measurable for fixed $z \in U$ and $z \mapsto f(t, z)$ is holomorphic for fixed $t \in X$. Prove that the function

$$\varphi(z) := \int_X f(t, z) d\mu(t)$$

is holomorphic in U .

F8. Let $f \in L^1[0, 1]$ and for $z \in \mathbb{C}$ define

$$g(z) := \int_0^1 f(t) e^{tz} dt.$$

If $g \equiv 0$, show that $f(t) = 0$ for almost every $t \in [0, 1]$.

F9. Let $u : \mathbb{D} \rightarrow \mathbb{R}$ be a positive harmonic function with $u(0) = 1$. How large or small can $u(1/2)$ be? Give the best possible bounds.

F10. Let $f : U \rightarrow \mathbb{C}$ be holomorphic in a domain U and consider an open topological disk D with $\bar{D} \subset U$. If $|f|$ is constant on the boundary of D and f has no zeros in D , prove that f is constant in U .

F11. Is there a sequence of polynomials $\{P_n\}$ in the complex plane such that $P_n(0) = 1$ for all n but $\lim_{n \rightarrow \infty} P_n(z) = 0$ for $z \neq 0$?

F12. Let $M > 0$ and $u : \mathbb{H} \rightarrow \mathbb{R}$ be a harmonic function satisfying

$$0 \leq u(x + iy) \leq My \quad \text{for all } x \in \mathbb{R} \text{ and all } y > 0.$$

Show that $u(x + iy) = ay$ for some constant $0 \leq a \leq M$.

F13. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a holomorphic map defined on the unit disk \mathbb{D} . For $r < 1$, define

$$M(r) := \sup_{|z|=r} |f(z)|, \quad M_1(r) := \sum_{n=0}^{\infty} |c_n| r^n, \quad M_2(r) := \left(\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \right)^{\frac{1}{2}}.$$

Fix $0 \leq r < R < 1$.

(a) Show that $M(r) \leq M_1(r) \leq \frac{R}{R-r} M(R)$.

(b) Show that $\frac{\sqrt{R^2 - r^2}}{R} M_1(r) \leq M_2(R) \leq M(R)$.

F14. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a holomorphic map defined on the unit disk \mathbb{D} . For $r < 1$, define

$$A(r) := \sup_{|z|=r} \operatorname{Re}(f(z)).$$

Prove that for every $r < 1$ and every $n \geq 0$,

$$|c_n| r^n + 2 \operatorname{Re}(f(0)) \leq 4 \max\{A(r), 0\}.$$

F15. (*Generalized Maximum Principle*) Let $U \subset \mathbb{C}$ be a bounded domain and $f : U \rightarrow \mathbb{C}$ be holomorphic. Assume that for every sequence $z_n \in U$ which converges to the boundary of U , we have $\limsup_{n \rightarrow \infty} |f(z_n)| \leq M$. Prove that $|f(z)| \leq M$ for every $z \in U$.

F16. Let $U \subset \mathbb{C}$ be a domain and $f_n : U \rightarrow \mathbb{C}$ be a sequence of holomorphic functions converging to a complex-valued function $f : U \rightarrow \mathbb{C}$ pointwise. Prove that there exists an open dense set $V \subset U$ on which f is holomorphic.

F17. Let $u : \mathbb{D}(0, R) \rightarrow \mathbb{R}$ be a harmonic function such that $|u| \leq M$ in $\mathbb{D}(0, R)$. Let v be the harmonic conjugate of u with $v(0) = 0$. Prove that

$$|v(z)| \leq \frac{2M}{\pi} \log \left(\frac{R+r}{R-r} \right)$$

if $|z| = r < R$. Find an example in which v is in fact unbounded in $\mathbb{D}(0, R)$.

F18. Let $u : \mathbb{D}(0, R) \rightarrow \mathbb{R}$ be a subharmonic function. For every $-\infty < t < \log R$, define

$$M(t) := \sup_{|z|=e^t} u(z).$$

- (a) Show that $M(t)$ is a convex function of t .
 (b) Show that any bounded subharmonic function $\mathbb{C} \rightarrow \mathbb{R}$ must be constant.

F19. Let $U \subset \mathbb{C}$ be a domain and $u_1, u_2, \dots, u_n : U \rightarrow \mathbb{C}$ be harmonic functions. Prove that the function

$$\varphi : z \mapsto |u_1(z)| + |u_2(z)| + \dots + |u_n(z)|$$

satisfies the Maximum Modulus Principle on U .

F20. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be a holomorphic map. Show that there exists a sequence $z_n \in \mathbb{D}$ with $|z_n| \rightarrow 1$ such that $\{f(z_n)\}$ is a bounded sequence in \mathbb{C} .

F21. Let $U \subset \mathbb{C}$ be a domain and $f, g : U \rightarrow \mathbb{C}$ be holomorphic. Assume $f(z) = f(w)$ if $g(z) = g(w)$. Show that there exists a holomorphic function $h : g(U) \rightarrow \mathbb{C}$ such that $f(z) = h(g(z))$ for all $z \in U$.

F22. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be holomorphic and assume $f'(z) \neq 0$ if $0 < |z| = r < 1$. Let Γ_r be the image of the circle $|z| = r$ under f . Find the distance from the origin to the line tangent to Γ_r at some $w = f(z)$.

F23. Under the conditions of problem F22, assuming that Γ_r has no self-intersection, show that Γ_r is a strictly convex closed curve if and only if whenever $|z| = r$,

$$\operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) > -1.$$

F24. Let \mathcal{F} be the family of all holomorphic functions $f : \mathbb{D} \rightarrow \mathbb{D}$ such that $f(0) = f'(0) = f''(0) = 0$. Show that \mathcal{F} contains a unique element g such that $g(1/2) = \sup_{f \in \mathcal{F}} |f(1/2)|$. Can you identify g explicitly?

G. Schwarz Lemma and Conformal Maps

G1. Let $P(z) = z + a_2z^2 + \dots + a_nz^n$ be univalent in the unit disk \mathbb{D} . Prove that $|a_n| \leq 1/n$, and show by an example that this is the best upper bound.

G2. (*Generalized Schwarz Lemma*) Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic and $f(z) = 0$ for $z = z_1, z_2, \dots, z_n$. Show that for all $z \in \mathbb{D}$,

$$|f(z)| \leq \prod_{k=1}^n \left| \frac{z - z_k}{1 - \overline{z_k}z} \right|.$$

G3. (*Real Bieberbach Conjecture*) Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be univalent, normalized as $f(0) = 0$ and $f'(0) = 1$ so that it has a power series expansion of the form $f(z) = z + \sum_{n=2}^{\infty} c_n z^n$. If c_n is real for all n , prove that $|c_n| \leq n$.

G4. Suppose that $f : \mathbb{H} \rightarrow \mathbb{C}$ is bounded and holomorphic, with $f(i) = 0$. How large or small can $|f(2i)|$ be? Show that your answers are the best possible estimates.

G5. Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic and $f(0) = 0$. Assume that for some real number $0 < r < 1$ we have $f(r) = f(-r) = 0$. Prove that

$$\left| f\left(\frac{r}{2}\right) \right| \leq \frac{3}{2} \left(\frac{r^3}{4 - r^4} \right).$$

G6. Find an explicit conformal map from the open half-disk $\{z \in \mathbb{C} : |z| < 1 \text{ and } \text{Im}(z) > 0\}$ onto the unit disk \mathbb{D} .

G7. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be holomorphic but not one-to-one. Prove that there exist $z, w \in \mathbb{D}$ such that $|z| = |w|$ and $f(z) = f(w)$.

G8. Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic. Prove that for every $z \in \mathbb{D}$,

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

G9. Let $f : \mathbb{D}(0, r) \rightarrow \mathbb{C}$ be holomorphic and $|f(z)| \leq M$ for all $z \in \mathbb{D}(0, r)$. Suppose that f vanishes at $z_1, z_2, \dots, z_n \in \mathbb{D}(0, r) \setminus \{0\}$. Prove that

$$\frac{r^n |f(0)|}{|z_1 z_2 \dots z_n|} \leq M.$$

G10. Let $f, g : \mathbb{D} \rightarrow \mathbb{C}$ be holomorphic, f be univalent, $f(0) = g(0) = 0$, and $g(\mathbb{D}) \subset f(\mathbb{D})$. Prove that for all $0 < r < 1$, $g(\mathbb{D}(0, r)) \subset f(\mathbb{D}(0, r))$ and hence $|g'(0)| \leq |f'(0)|$.

G11. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be holomorphic, $|\text{Re}(f)| < 1$ in \mathbb{D} and $f(0) = 0$. Prove that for all $r < 1$,

$$|f(re^{i\theta})| \leq \frac{2}{\pi} \log \left(\frac{1+r}{1-r} \right).$$

G12. For which values of $\lambda \in \mathbb{C}$ is the quadratic polynomial $Q_\lambda(z) = \lambda z + z^2$ univalent in the unit disk?

G13. Let ω be a primitive k -th root of unity and $\Omega \subset \mathbb{C}$ be a simply-connected domain such that $\omega\Omega = \Omega$, i.e., $z \in \Omega$ if and only if $\omega z \in \Omega$. Let $f : \mathbb{D} \rightarrow \Omega$ be a univalent map with $f(0) = 0$, with power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Prove that $c_n = 0$ for every n such that $n \not\equiv 1 \pmod{k}$.

G14. Let $U \subset \mathbb{C}$ be a domain with more than one boundary point, $a \in U$, and $f : U \rightarrow U$ be a holomorphic map with $f(a) = a$.

- (a) Show that $|f'(a)| \leq 1$.
- (b) If $|f'(a)| = 1$, what can you say about f ?
- (c) If $f'(a) = 1$, what can you say about f ?
- (d) Do these results hold when $U = \mathbb{C}$ or $U = \mathbb{C} \setminus \{\text{point}\}$?

G15. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be univalent and for some $0 < r < 1$ the image $f(\mathbb{D}(0, r))$ be convex. Prove that for every $0 < s < r$, the image $f(\mathbb{D}(0, s))$ is also convex.

G16. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be univalent, normalized as $f(0) = 0$ and $f'(0) = 1$. For $r < 1$, let V_r denote the area of the Jordan domain $f(\mathbb{D}(0, r))$. Prove that $V_r \geq \pi r^2$.

G17. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be univalent and $0 \notin f(\mathbb{D})$. Show that

- (a) $|f'(0)| \leq 4|f(0)|$.
- (b) for all $z \in \mathbb{D}$, $\left(\frac{1-|z|}{1+|z|}\right)^2 \leq \frac{|f(z)|}{|f(0)|} \leq \left(\frac{1+|z|}{1-|z|}\right)^2$.

H. Entire Maps and Normal Families

H1. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire map which satisfies

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq r^{\frac{17}{3}}$$

for all $r > 0$. Prove that $f \equiv 0$.

H2. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an entire map which satisfies

$$|f(z)| \leq |\operatorname{Re}(z)|^{-\frac{1}{2}}$$

for all z off the imaginary axis. Prove that $f \equiv 0$.

H3. Let $f, g : \mathbb{C} \rightarrow \mathbb{C}$ be entire maps with no common zeros. Prove that there exist entire maps F, G such that for all $z \in \mathbb{C}$,

$$f(z)F(z) + g(z)G(z) = 1.$$

H4. Let $f, g : \mathbb{C} \rightarrow \mathbb{C}$ be non-constant entire maps with $f(0) = 0$ and $g(0) = 1$. Prove that there are infinitely many points $z \in \mathbb{C}$ at which $|f(z)| = |g(z)|$.

H5. Prove that the exponential function $z \mapsto e^z$ has infinitely many fixed points in the complex plane.

H6. Let $A, B : U \rightarrow \mathbb{C}$ be holomorphic functions defined in a domain U such that $A(z) \neq B(z)$ for all $z \in U$. Let $f_n : U \rightarrow \mathbb{C}$ be a sequence of holomorphic functions such that $f_n(z) \neq A(z)$ and $f_n(z) \neq B(z)$ for all n and all $z \in U$. Prove that $\{f_n\}$ is a normal family.

H7. Fix $M > 0$ and let \mathcal{F} be the class of all holomorphic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 dx dy \leq M.$$

Is \mathcal{F} a normal family?

H8. Let $U := \mathbb{D} \setminus \{0\}$ and $f : U \rightarrow \mathbb{C}$ be holomorphic with an essential singularity at $z = 0$. For $n \geq 1$ and $z \in U$ define $f_n(z) := f(2^{-n}z)$. Can the sequence $\{f_n\}$ be normal on U ?

H9. Let $f, g : \mathbb{C} \rightarrow \mathbb{C}$ be entire maps with $e^f + e^g \equiv 1$. Show that f and g are both constants.

H10. Does there exist a non-constant entire map $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $\operatorname{Re}(f(z)) \neq (\operatorname{Im}(f(z)))^2$ for all z ?

H11. Let $\sum_{n=0}^{\infty} a_n z^n$ be holomorphic in \mathbb{D} with $a_n \geq 0$ for all n . Let \mathcal{F} denote the family of all holomorphic maps $f(z) = \sum_{n=0}^{\infty} c_n z^n$ on the unit disk such that $|c_n| \leq a_n$ for all n . Prove that \mathcal{F} is normal.

H12. Let $f, g : \mathbb{C} \rightarrow \mathbb{C}$ be entire maps with $f(0) = g(0)$, and let $P, Q : \mathbb{C} \rightarrow \mathbb{C}$ be polynomials. Assume $e^{f(z)} + P(z) = e^{g(z)} + Q(z)$ for all $z \in \mathbb{C}$. Show that $f \equiv g$ and hence $P \equiv Q$.

II. Answers and Hints

A. Basic Real Analysis

- A1.** A bounded polynomial on \mathbb{R} must be constant.
- A2.** g maps every fixed point of f to a fixed point of f .
- A3.** For $x, y \in X$ and $\varepsilon > 0$, consider the sequence of iterates $x_n := f^{\circ n}(x)$ and $y_n := f^{\circ n}(y)$ and show that $d(x, x_n) < \varepsilon$ and $d(y, y_n) < \varepsilon$ for some $n \geq 1$.
- A4.** Consider the function $t \mapsto t - t^2$ for real t .
- A5.** Every uncountable closed subset of the circle contains a perfect set.
- A6.** The condition of the problem is symmetric with respect to 2 or 3.
- A7.** Use the fact that $|f(x)| \leq x \sup_{0 \leq t \leq x} |f(t)|$.
- A8.** Every $n \in I \cap [10^k, 10^{k+1})$ can be written as $a10^k + b$, where $1 \leq a \leq 8$ and $b \in I \cap [10^s, 10^{s+1})$ for some $0 \leq s \leq k - 1$, or $b = 0$. This allows you to estimate the sum inductively.
- A9.** Look at the graph of the logarithmic derivative P'/P .
- A10.** There is a unique n with the property $x_n = x_{n+1}$. Find this n .
- A11.** Form the characteristic equation to solve the recursion explicitly.
- A12.** If $\sum_{n=1}^{\infty} |a_n|^2 = \infty$, consider the partition of \mathbb{N} into consecutive finite segments E_1, E_2, E_3, \dots such that $\sum_{n \in E_k} |a_n|^2 > 1$. Define a sequence $\{b_n\}$ by setting $b_n := c_k a_n$ for $n \in E_k$. Choose c_k such that $\sum_{n=1}^{\infty} |b_n|^2 < \infty$ but $\sum_{n=1}^{\infty} |a_n b_n| = \infty$.
- A13.** The answer is no. Use, for example, Baire's Category Theorem to obtain a contradiction to the existence of such a partition.
- A14.** Look at the function $M(t) = \sup_{0 \leq x \leq t} f(x)$ for $t \geq 0$.
- A15.** Use Baire's Category Theorem.
- A16.** Represent I as a double integral over the unit square.

B. Measures and Integrals

- B1.** Consider the characteristic functions of the E_i .
- B2.** Use, for example, Lebesgue's Differentiation Theorem.
- B3.** To prove that f_k has no limit at any point, use the fact that for any $x \in [0, 1]$ and any integer $q > 0$, there exists an integer p between 0 and q such that $|x - p/q| \leq 1/q$.
- B4.** f is the supremum of a collection of measurable functions. Show that $f = 9$ almost everywhere.
- B5.** $m(E) = 2\pi/3$.
- B6.** The answer is no. Use the fact that every measurable set of positive measure in

\mathbb{R} has non-measurable subsets.

B7. Consider the partition of $[0, 1]$ into dyadic intervals of length 2^{-k} and let $0 \leq n_k(y) \leq 2^k$ be the number of such intervals containing a solution of $f(x) = y$. Show that $n(y) = \lim_{k \rightarrow \infty} n_k(y)$.

B8. Define $E_{n,q}$ as the set of x such that $|x - p/q| < 1/(nq^\nu)$ for some integer $0 \leq p \leq q$. Find the relation between D_ν and the $E_{n,q}$.

B9. Use Borel-Cantelli to show that $m(E) = 0$.

B10. $m(F) = 0$. To prove this, assume otherwise and consider density points for E and F .

B11. $a = \text{area}(K)$, $b = \text{length}(\partial K)$, and $c = \pi$.

B12. Let $X_n(\varepsilon)$ denote the set of points x where $|f_n(x) - f(x)| > \varepsilon$, and split the expression for $\rho(f_n - f)$ into two integrals taken over $X_n(\varepsilon)$ and $X \setminus X_n(\varepsilon)$.

B13. Use the fact that $G(x) = 1/x - k$ if $1/(k+1) < x \leq 1/k$. Take an interval and express its G -preimage as a union of disjoint intervals. Calculate measures explicitly.

B14. This space is not separable. Try to find an uncountable family of functions in $L^\infty[0, 1]$ in which the mutual distance between any two members is larger than a definite constant.

B15. For the first equality, use the fact that for every $\varepsilon > 0$, p_n is larger than the integral of $|f|^n$ over the set of points where $|f(x)| > \|f\|_\infty - \varepsilon$. For the second equality, use Hölder Inequality and the first equality.

B16. Use the fact that continuous functions with compact support are dense in $L^p(\mathbb{R})$.

B17. If f is M -Lipschitz, it has a derivative $f' \in L^1[0, 1]$ and the Fundamental Theorem of Calculus holds for f . Use Lusin's Theorem to approximate f' by continuous functions and from there define the sequence f_n .

B18. Use Fatou's Lemma and Hölder Inequality to show that $f \in L^1$. Use Egoroff's Theorem to prove the second part.

B19. Define X_n as the set of x where $|f(x)|$ lies between $\|f\|_\infty/2^n$ and $\|f\|_\infty/2^{n-1}$. Write the integral of f over X as the sum of the integrals of f over the X_n .

B20. Apply Borel-Cantelli to the sets $\{x : |f_n(x)| > \alpha_n\}$.

B21. Use the fact that E has density points.

B22. For (a), define X_n as the set of x where $|f(x)| \geq 2^n$. Decompose X using the difference sets $X_n \setminus X_{n+1}$ and integrate. For (b), define X_n as the set of x where $|f(x)| \geq 1/2^n$ and proceed in the same way.

B23. Use the fact that every real symmetric matrix can be diagonalized by an orthogonal transformation.

B24. For $t > 0$, define E as the set of x where $\sum |c_n f_n(x)| > t$. Show that

$\sum |c_n| \leq t/(A - B\sqrt{m(E)})$ for large t .

B25. Use the Riesz Representation Theorem.

B26. If $\varphi \notin L^\infty(\Omega)$, every set $E_n = \{x \in \Omega : |\varphi(x)| > n\}$ has positive measure. Find a suitable function f , constant on every difference set $E_n \setminus E_{n+1}$, such that $f \in L^p(\Omega)$ but $\varphi f \notin L^p(\Omega)$.

B27. Use Baire's Category Theorem.

C. Banach and Hilbert Spaces

C1. Use the Riesz Representation Theorem.

C2. Perturb T slightly to make it into a contraction and apply the Contraction Mapping Principle.

C3. Use the inequality $|f(x)| \leq \|f\|\|x\|$ as well as the definition of the norm $\|f\|$.

C4. E is Banach. Use the fact that $\|x + F\| \leq \|x\|$ for all $x \in E$ and if $\|x_n + F\| \rightarrow 0$, then $\|y_n\| \rightarrow 0$ for some $y_n \in x_n + F$.

C5. Apply Closed Graph Theorem. For a counterexample when H is not complete, work in the space of C^∞ functions on the real line which vanish outside the unit interval.

C6. The answer is no. Work in the space $C[-1, 1]$, taking F to be the subspace of all continuous functions f with $\int_{-1}^0 f = \int_0^1 f$.

C7. First show that there exists a constant $C > 0$ such that $|\langle Tx, x \rangle| \geq C\|x\|^2$ for all $x \in H$.

C8. Consider $\sup_{n \in \mathbb{Z}} \|T^n x\|$.

C9. To show T_n is monic in (a), use complex numbers and the exponentiation formula. The distance in (b) is $1/2^{n-1}$.

C10. Use the Riesz Representation Theorem and the definition of norm.

C11. The answer is no. Assuming the existence of such a norm, construct a sequence of functions in $C[0, 1]$ of unit norm which converges pointwise to zero.

D. Fourier Series and Integrals

D1. f is necessarily a linear combination of $\sin x$ and $\cos x$. Use the fact that the n -th Fourier coefficient of f' is $in\hat{f}(n)$, and apply Parseval's Identity.

D2. When does equality occur in Triangle Inequality?

D3. Without loss of generality assume $\|f'\| \equiv 1$ to simplify the argument. Write the Fourier series of the coordinate functions of f , and apply Green's Theorem and Parseval's Identity to estimate A .

D4. Use the Fourier series of the characteristic function of E .

- D5.** Use the old trick $\hat{f}(n) = -1/(2\pi) \int_0^{2\pi} f(x + \pi/n)e^{-inx} dx$.
- D6.** Use Integration by Parts and Riemann-Lebesgue Lemma.
- D7.** Use the fact that the Fourier series of every L^2 function f has a sub-series which converges almost everywhere to f . (This is a soft L^p result; a much harder theorem due to L. Carleson asserts that the entire series converges almost everywhere to f , but you do not need this difficult result here.)
- D8.** Either use the Fourier series of f and Δf , or consider $g := f^2$ and its Laplacian and apply Green's Theorem to integrate Δg over the boundary of the unit square.
- D9.** Write the Fourier series of f_ε and use Parseval's Identity.
- D10.** Recall the construction of trigonometric polynomials which uniformly approximate a given continuous function, or apply the Theorem of Fejér.
- D11.** Use the Theorem of Fejér.
- D12.** Estimate $|\hat{f}(0)|$ directly and use Parseval's Identity and Cauchy-Schwarz to estimate $\sum_{n \neq 0} |\hat{f}(n)|$.

E. Basic Complex Analysis

- E1.** Use Cauchy-Schwarz Inequality.
- E2.** Write $z_k := r_k e^{i\theta_k}$ with $0 \leq \theta_k < 2\pi$. Let I_θ be the set of integers k between 1 and n such that $\cos(\theta - \theta_k) \geq 0$. Show that the required index set I can be taken as I_{θ_0} , where θ_0 is an angle which maximizes the sum $\sum_{k=1}^n r_k \cos^+(\theta - \theta_k)$.
- E3.** All the roots of P are on the unit circle, and $x = 1$ is one of them.
- E4.** Integrate $1 - \sum z_k e^{2\pi ikt}$ from 0 to 1.
- E5.** g is a polynomial.
- E6.** Apply the Residue Theorem to $1/P$.
- E7.** Use the fact that if $P(z) + P^*(z) = 0$ for some z , then $|P(z)| = |P^*(z)|$.

F. Properties of Holomorphic and Harmonic Functions

- F1.** Use Cauchy's Formula and integrate over a circle of radius $0 < r < 1 - |z|$ centered at z . Then integrate with respect to r .
- F2.** If $f(z) = \sum c_n z^n$, express c_n in terms of some integral of f' over the circle $|z| = r$ to estimate the growth of c_n . Then use this estimate to prove the result.
- F3.** Use the local behavior of holomorphic maps near a critical point.
- F4.** Use the power series of e^z .
- F5.** Use the fact that u is the real part of an entire map, and apply Cauchy's estimates.
- F6.** Use the Inverse Function Theorem.

F7. First show that for every compact set $K \subset U$ there exists an $M = M(K) > 0$ such that $|f(t, z) - f(t, w)|/|z - w| \leq M$ whenever z, w are distinct points of K and $t \in X$.

F8. First show that $\int_0^1 t^n f(t) dt = 0$ for all $n \geq 0$.

F9. Use Harnack's Inequality or Schwarz Lemma to show that $1/3 \leq u(1/2) \leq 3$.

F10. Use the Maximum Principle.

F11. Use the Theorem of Runge on uniform approximation of holomorphic maps by polynomials to construct such a sequence.

F12. Extend u to the complex plane by symmetry and show that the entire map whose real part is this extension has bounded derivative.

F13. For (a), use Triangle Inequality and Cauchy's estimates. For (b), use Cauchy-Schwarz Inequality and Parseval's Identity.

F14. First show that $c_n r^n = (1/\pi) \int_0^{2\pi} \operatorname{Re}(f)(re^{i\theta})e^{-in\theta} d\theta$.

F15. Use the ordinary Maximum Principle to prove $\{z : |f(z)| > M + \varepsilon\}$ is empty for every $\varepsilon > 0$.

F16. Define $\varphi(z) := \sup_n |f_n(z)|$. Show that every disk in U has a subdisk on which φ is uniformly bounded.

F17. Apply Poisson's Formula to represent v as an integral involving u . Then estimate the integral directly. To find a counterexample, map the disk $\mathbb{D}(0, R)$ conformally to a vertical strip.

F18. For (a), compare u to the harmonic function $z \mapsto \alpha \log |z| + \beta$ for suitable constants α and β and use Maximum Principle. For (b), use the fact that a bounded convex function on $(-\infty, +\infty)$ is constant.

F19. Show that φ is subharmonic.

F20. It suffices to consider the case where f has finitely many zeros in \mathbb{D} . Assuming that no such sequence $\{z_n\}$ exists, divide f by a Blaschke product with the same zeros as f , and study the resulting function to obtain a contradiction.

F21. Construct h locally away from the critical values of g . Show that these values are removable singularities for h .

F22. The required distance is $\operatorname{Re}(zf'(z)\overline{f'(z)})/|zf'(z)|$.

F23. Show that the curvature of Γ_r at a point $w = f(z)$ is given by the formula $\kappa = [1 + \operatorname{Re}(zf''(z)/f'(z))]/|zf'(z)|$.

F24. Show that $g(z)$ must be z^3 .

G. Schwarz Lemma and Conformal Maps

G1. Use the fact that all the critical points of P are outside \mathbb{D} so their product is at least 1 in absolute value.

G2. Divide f by an appropriate Blaschke product and apply the Maximum Principle.

G3. First show that for every $0 < r < 1$, $c_n = \frac{1}{\pi r^n} \int_0^{2\pi} \operatorname{Im}(f)(re^{i\theta}) \sin n\theta \, d\theta$.

G4. Use Schwarz Lemma to show that $0 \leq |f(2i)| \leq M/3$, where $M > 0$ is any upper bound for $|f|$ on \mathbb{H} .

G5. Use problem G2.

G6. First map the open half-disk to the first quadrant using a Möbius transformation.

G7. Use the Argument Principle.

G8. Use appropriate disk automorphisms to reduce the problem to the standard Schwarz Lemma.

G9. Use problem G2.

G10. Apply Schwarz Lemma to $f^{-1} \circ g$.

G11. Find a conformal map from the strip $-1 < \operatorname{Re}(z) < 1$ to the unit disk and apply Schwarz Lemma.

G12. Q_λ is univalent in \mathbb{D} if and only if $|\lambda| \geq 2$.

G13. Show that $f(\omega z) = \omega f(z)$.

G14. Use the Uniformization Theorem.

G15. Use problem G10, or problem F23.

G16. Express V_r as an integral of $|f'|^2$.

G17. For (a), use Koebe 1/4-Theorem. For (b), precompose f with an appropriate Möbius transformation of the disk, apply (a), and then integrate the resulting inequality.

H. Entire Maps and Normal Families

H1. Use Cauchy's Formula to show that $f^{(n)}(0) = 0$ for all $n \geq 0$.

H2. Use Cauchy's Formula to represent $f^{(n)}(0)$ as an integral over a large square centered at the origin.

H3. Use the Theorem of Mittag-Leffler to show that there exists an entire function F such that at every root of g , the function $(1 - fF)$ vanishes at a higher order than g .

H4. First show that $\{z : |f(z)| > |g(z)|\}$ is non-empty.

H5. Use Picard's Theorem.

H6. Consider $(f_n - A)/(f_n - B)$ and apply Montel's Theorem.

H7. The answer is yes. Show that $|f(z)| \leq (\text{const.})/(1 - |z|^2)$ for $f \in \mathcal{F}$ and $z \in \mathbb{D}$. Then apply Montel's Theorem.

H8. Show that the assumption of normality would lead to 0 being a pole of f .

H9. Use Picard's Theorem.

H10. The answer is no. Use Picard's Theorem.

H11. Apply Montel's Theorem.

H12. Write $P - Q = e^g(1 - e^{f-g})$ and apply Picard's Theorem.

III. Solutions

A. Basic Real Analysis

A1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the uniform limit of a sequence $\{P_n\}$ of polynomials. Since $\{P_n\}$ is a Cauchy sequence, there exists an $N > 0$ such that $\sup_{x \in \mathbb{R}} |P_n(x) - P_N(x)| < 1$ if $n \geq N$. This implies that $P_n - P_N$ is a bounded polynomial on \mathbb{R} , so $P_n = c_n + P_N$, where $c_n \in \mathbb{R}$. Hence $f = \lim_{n \rightarrow \infty} P_n = (\lim_{n \rightarrow \infty} c_n) + P_N = c + P_N$, where $c = \lim_{n \rightarrow \infty} c_n$. Therefore f itself must be a polynomial.

A2. Let $\text{Fix}(f)$ denote the set of fixed points of f , and similarly define $\text{Fix}(g)$. An easy calculus exercise shows that these are closed non-empty sets. Since $f \circ g = g \circ f$, we have $g(\text{Fix}(f)) \subset \text{Fix}(f)$. Let $\alpha := \sup \text{Fix}(f)$. If $g(\alpha) = \alpha$, we are done. If not, $g(\alpha) < \alpha$ and hence $\{g^{on}(\alpha)\}$ is a decreasing sequence in $\text{Fix}(f)$. Hence $\beta := \lim_{n \rightarrow \infty} g^{on}(\alpha)$ is a common fixed point of f and g .

A3. Fix $x, y \in X$ and $\varepsilon > 0$. Consider $x_n := f^{on}(x)$ and $y_n := f^{on}(y)$. Since X is compact, there exist $k > m > 1$ such that $d(x_k, x_m) < \varepsilon$ and $d(y_k, y_m) < \varepsilon$. If $n := k - m$, it follows from the expanding condition on f that

$$d(x_n, x) < \varepsilon \text{ and } d(y_n, y) < \varepsilon. \quad (1)$$

The Triangle Inequality then shows

$$d(x_1, y_1) \leq d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) \leq 2\varepsilon + d(x, y).$$

Letting $\varepsilon \rightarrow 0$, we get $d(f(x), f(y)) \leq d(x, y)$. Hence we have the equality

$$d(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$. But $f(X)$ is dense in X by (1), and is compact since f is continuous. Therefore, $f(X) = X$.

A4. Note that $t - t^2 < 1/4$ for all real $t \neq 1/2$. Hence if $f(0) \neq 1/2$, then $c = 0$ satisfies $f(c^2) - (f(c))^2 < 1/4$. If $f(0) = 1/2$, then $f(1) \neq 1/2$ since f is one-to-one. Hence $c = 1$ satisfies the required inequality.

A5. We show the existence of such a set as follows: Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational angles on \mathbb{T} . For each n , remove from \mathbb{T} an open arc I_n of length $< 2^{-(n+1)}$ containing r_n . Then $\mathbb{T} \setminus \bigcup_{n=1}^{\infty} I_n$ is closed, and is non-empty because it has measure $> 1/2$. By the Theorem of Cantor-Bendixson ([R1], Exercise 2.28), this

set is a union of an at most countable set and a perfect set E . Clearly E does not contain any rational angle.

A6. First note that with probability 1, both digits 2 and 3 appear in the decimal expansion of x . To see this, let A_n be the set of points $x = 0.x_1x_2x_3\cdots$ with $x_j = 2$ for some $1 \leq j \leq n$. An easy exercise shows that $m(A_n) = \sum_{j=1}^n 9^{j-1}10^{-j}$. Hence the set of all x with a digit 2 somewhere in their decimal expansion has measure $\lim_{n \rightarrow \infty} m(A_n) = \sum_{j=1}^{\infty} 9^{-1}0.9^j = 1$. One can prove the similar statement for the digit 3 instead of 2.

Now the property of 2 appearing before 3 is symmetric with respect to 2 and 3, so the required probability must be $1/2$. (Those who do not find this argument “rigorous” enough are invited to compute the measures directly!)

A7. If $f \not\equiv 0$, take an $a \in (0, 1)$ such that $f(a) \neq 0$ and choose $x \in (0, a]$ such that $|f(x)| = \sup_{0 \leq t \leq a} |f(t)|$. Clearly $|f(x)| > 0$ and $x < 1$. Then

$$|f(x)| \leq \int_0^x f(t) dt \leq \int_0^x |f(t)| dt \leq x \sup_{0 \leq t \leq x} |f(t)| \leq x \sup_{0 \leq t \leq a} |f(t)| = x|f(x)|,$$

which contradicts $x < 1$.

A8. Let $I_k := I \cap [10^k, 10^{k+1})$ and $S_k := \sum_{n \in I_k} 1/n$. For a fixed $k \geq 1$, each $n \in I_k$ has the decimal representation

$$\overline{n_k n_{k-1} \cdots n_1 n_0}, \quad n_i \neq 9 \text{ for } 0 \leq i \leq k \text{ and } n_k \neq 0.$$

Hence $n = 10^k n_k + j$, where $1 \leq n_k \leq 8$ and $j \in I_s$ for some $0 \leq s \leq k-1$ or else $j = 0$. It follows in particular that $\#I_k = 8 \sum_{s=0}^{k-1} \#I_s + 8$, which implies by an induction that $\#I_k = 8 \cdot 9^k$. Thus

$$\begin{aligned} S_k &= \sum_{i=1}^8 \sum_{s=0}^{k-1} \sum_{j \in I_s} \frac{1}{10^k i + j} + \sum_{i=1}^8 \frac{1}{10^k i} \\ &\leq 8 \sum_{s=0}^{k-1} \sum_{j \in I_s} \frac{1}{10^k} + \frac{8}{10^k} \\ &= 10^{-k} \left(8 \sum_{s=0}^{k-1} \#I_s + 8 \right) \\ &= 10^{-k} \#I_k \\ &< 8(0.9)^k. \end{aligned}$$

This implies

$$\sum_{n \in I} \frac{1}{n} = S_0 + \sum_{k=1}^{\infty} S_k \leq S_0 + 8 \sum_{k=1}^{\infty} (0.9)^k = S_0 + 72 < 75.$$

A9. Assume $r \neq 0$ and let $P(x) = C(x - a_1)^{n_1} \cdots (x - a_k)^{n_k}$, where C and the a_j are distinct real numbers and $n_j \geq 1$. Clearly when $n_j > 1$, a_j is a root of Q_r of multiplicity $n_j - 1$. On the other hand, when x is distinct from the roots a_j , the equation $Q_r(x) = 0$ is equivalent to $P'(x)/P(x) = 1/r$, or $f(x) := \sum_{j=1}^k n_j/(x - a_j) = 1/r$. By considering the graph of f , one easily sees that the equation $f(x) = 1/r$ has exactly k real roots $\neq a_j$ for each $r \neq 0$. It follows that Q_r has at least $k + \sum_{j=1}^k (n_j - 1) = \sum_{j=1}^k n_j = \deg P$ real roots. Since $\deg P = \deg Q_r$, it follows that these must form all the roots of Q_r .

A10. It is easy to see that $x_{999} = x_{1000}$. Since

$$x_{n+1} = \frac{1000}{n+1} x_n,$$

it follows that $\{x_n\}_{1 \leq n \leq 999}$ is increasing and $\{x_n\}_{n \geq 1000}$ is decreasing. We conclude that $\max_{n \geq 1} x_n = x_{1000} = 1000^{1000}/1000!$.

A11. The recursion can be written as

$$x_n - 2rx_{n-1} + x_{n-2} = 0, \quad n \geq 2.$$

The characteristic equation of this recursion is $\lambda^2 - 2r\lambda + 1 = 0$, which has roots $\alpha, \beta = r \pm \sqrt{r^2 - 1}$. If $r \neq \pm 1$, we have the general solution

$$x_n = A\alpha^n + B\beta^n, \quad n \geq 0. \quad (2)$$

On the other hand, if $r = \pm 1$, then $\alpha = \beta = r$ and the general solution will be

$$x_n = (A + Bn)r^n, \quad n \geq 0. \quad (3)$$

The constants A and B are determined by the initial conditions x_0, x_1 . We distinguish three cases:

- If $r > 1$ or $r < -1$, then at least one of the real roots α or β has absolute value bigger than 1. Hence by (2) the sequence $\{x_n\}$ will be unbounded (by a suitable choice of x_0, x_1), so it cannot be periodic.

- Similarly, if $r = \pm 1$, (3) shows that $\{x_n\}$ cannot be periodic.

- If $-1 < r < 1$, let $r = \cos t$. Then from (2) it follows that

$$x_n = Ce^{int} + De^{-int}$$

for some constants C, D . The condition $x_n = x_{n+p}$ implies that $t = 2k\pi/p$ for some $k \in \mathbb{Z}$. Hence $r = \cos(2k\pi/p)$ for $k = 1, 2, \dots, p-1$. It is easy to check that every such r satisfies the required condition.

A12. Assume $\sum_{n=1}^{\infty} |a_n|^2 = \infty$. Consider the partition of \mathbb{N} into consecutive finite segments E_1, E_2, E_3, \dots such that $A_k := \sum_{n \in E_k} |a_n|^2 > 1$. Define a sequence $\{b_n\}$ by setting

$$b_n := \frac{1}{k\sqrt{A_k}} a_n \quad \text{if } n \in E_k.$$

Then

$$\sum_{n=1}^{\infty} |b_n|^2 = \sum_{k=1}^{\infty} \sum_{n \in E_k} |b_n|^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

while

$$\sum_{n=1}^{\infty} |a_n b_n| = \sum_{k=1}^{\infty} \sum_{n \in E_k} |a_n b_n| = \sum_{k=1}^{\infty} \frac{\sqrt{A_k}}{k} > \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

A13. *First solution.* The answer is negative. Assume by way of contradiction that $[0, 1] = \bigcup_{n=1}^{\infty} E_n$, where the E_n are non-empty, closed and disjoint. Since E_1 and E_2 are disjoint, there exists an open neighborhood U_2 of E_2 whose closure \overline{U}_2 is disjoint from E_1 . Note that U_2 is a countable union of disjoint open intervals, at least one of which intersects E_2 . Let C_2 be the closure of such an interval. It is easy to see that the closed interval C_2 cannot be contained in E_2 , and since C_2 intersects E_2 , it cannot be contained in any of the E_n . In brief,

$$E_1 \cap C_2 = \emptyset \quad \text{and} \quad C_2 = \bigcup_{n=2}^{\infty} (E_n \cap C_2).$$

Now apply the above argument to the interval C_2 (instead of $[0, 1]$) and the disjoint closed sets $E_n \cap C_2$ (instead of E_n). It follows that there exists a closed interval C_3 such that

$$C_2 \supset C_3 \quad \text{and} \quad E_2 \cap C_2 \cap C_3 = \emptyset \quad \text{and} \quad C_2 \cap C_3 = \bigcup_{n=3}^{\infty} (E_n \cap C_2 \cap C_3).$$

Continuing inductively, we find a nested sequence $C_2 \supset C_3 \supset C_4 \supset \dots$ of closed intervals such that for all $p \geq 2$,

$$E_{p-1} \cap \bigcap_{k=2}^p C_k = \emptyset \quad \text{and} \quad \bigcap_{k=2}^p C_k = \bigcup_{n=p}^{\infty} \left(E_n \cap \bigcap_{k=2}^p C_k \right).$$

Clearly $\bigcap_{k=2}^{\infty} C_k \neq \emptyset$. On the other hand,

$$\bigcap_{k=2}^{\infty} C_k = \left(\bigcup_{n=1}^{\infty} E_n \right) \cap \left(\bigcap_{k=2}^{\infty} C_k \right) = \bigcup_{n=1}^{\infty} \left(E_n \cap \bigcap_{k=2}^{\infty} C_k \right) = \emptyset,$$

which is a contradiction.

Second solution. Again, consider a partition $[0, 1] = \bigcup_{n=1}^{\infty} E_n$ as above. By Baire's Category Theorem, some E_n must contain an open interval, which we may assume is maximal in E_n . Remove this maximal interval from $[0, 1]$ and repeat the process with what is left. We end up with a countable collection $\{I_k\}$ of disjoint open intervals, each being maximal in some E_n , such that $\bigcup_{k=1}^{\infty} I_k$ is dense in $[0, 1]$. Note that no two of these intervals can have a common endpoint (since the E_n are disjoint). Consider the closed set $F := [0, 1] \setminus \bigcup_{k=1}^{\infty} I_k$. It is easy to see that F is perfect (i.e., has no isolated point) and it consists of all the endpoints of the I_k and all the points accumulated by such endpoints. Since $F = \bigcup_{n=1}^{\infty} (F \cap E_n)$, one can apply Baire's Category Theorem this time to F to conclude that some $F \cap E_n$ has interior in F , i.e., there exists an index n and an open interval J such that $F \cap E_n \supset F \cap J \neq \emptyset$. In particular, if the endpoints of some I_k belong to J , then $I_k \subset E_n$. Since $\bigcup_{k=1}^{\infty} I_k$ is dense, it follows that $J \subset E_n$, meaning that E_n has interior. Since the maximal interval in the interior of E_n which contains J is one of the I_k (and hence is disjoint from F), it follows that $F \cap J = \emptyset$, which is a contradiction.

A14. *First solution.* For $t \geq 0$ define $M(t) := \sup_{0 \leq x \leq t} f(x)$. Then M is continuous, non-decreasing, $M(0) = f(0)$, and $M(t) \geq f(t)$ for all $t \geq 0$. Since f has a local minimum at every point, it is easy to see that if $M(t) = f(t)$ for some $t > 0$, then $f(x) = f(0)$ for $0 \leq x \leq t$. Set $\alpha := \sup\{t \geq 0 : M(t) = f(t)\}$ and assume that $\alpha < +\infty$. Then $M(\alpha) = f(\alpha)$ since both M and f are continuous, $f(x) = f(0)$ for $0 \leq x \leq \alpha$ and $M(t) > f(t)$ for $t > \alpha$. It follows that the maximum of f on $[\alpha, \alpha + 1]$ occurs at some $\beta \in (\alpha, \alpha + 1)$. But this implies $M(\beta) = f(\beta)$, which contradicts the definition of α . Therefore, $\alpha = +\infty$ and f is constant for $x \geq 0$. A similar argument proves that f is constant for $x \leq 0$.

Second solution. Let \mathbb{R}_ℓ denote the real line equipped with the topology generated by the intervals of the form $a \leq x < b$. Then $f : \mathbb{R} \rightarrow \mathbb{R}_\ell$ is continuous. But \mathbb{R} is connected and \mathbb{R}_ℓ is totally disconnected, hence f must be constant.

A15. Fix $\varepsilon > 0$ and for each $k \geq 1$ define

$$E_k := \{x \in [1, 2] : |f(nx)| \leq \varepsilon \text{ for all } n \geq k\}.$$

Evidently each E_k is closed and $[1, 2] = \bigcup_{k=1}^{\infty} E_k$. Therefore, by Baire's Category Theorem, we can find a k and an open interval J such that $J \subset E_k$. It is easy to see

that the union $\bigcup_{n=k}^{\infty} (nJ) := \{y : y = nx \text{ for some } x \in J \text{ and some } n \geq k\}$ contains an interval of the form $(R, +\infty)$. It follows that for every $y > R$, $|f(y)| \leq \varepsilon$.

A16. Note that

$$I = \int \int_S \frac{f(x)}{f(y)} dx dy = \int \int_S \frac{f(y)}{f(x)} dx dy,$$

where $S = [0, 1] \times [0, 1]$ is the unit square in the plane. It follows that

$$I = \frac{1}{2} \int \int_S \left(\frac{f(x)}{f(y)} + \frac{f(y)}{f(x)} \right) dx dy = \int \int_S \frac{f^2(x) + f^2(y)}{2f(x)f(y)} dx dy.$$

Since $f^2(x) + f^2(y) \geq 2f(x)f(y)$, the last integrand is bounded below by 1. Hence $I \geq 1$.

B. Measures and Integrals

B1. Let χ_i denote the characteristic function of E_i . Then $\sum_{i=1}^n \chi_i \geq q$, which by integrating gives $\sum_{i=1}^n m(E_i) \geq q m[0, 1] = q$. This implies that at least one of the terms $m(E_i)$ must be $\geq q/n$.

B2. First solution. We show that the given inequality for intervals J can be generalized to all measurable sets $E \subset [0, 1]$, i.e.,

$$0 \leq \int_E f(x) dx \leq m(E). \tag{4}$$

In fact, one can write a given open set $E \subset [0, 1]$ as the disjoint union of countably many open intervals J_i . Then

$$0 \leq \int_E f(x) dx = \sum_i \int_{J_i} f(x) dx \leq \sum_i m(J_i) = m(E),$$

so (4) holds when E is open. Now, by regularity of Lebesgue measure, given any measurable set E one can find a sequence of open sets O_n containing E such that $m(O_n \setminus E) \rightarrow 0$ as $n \rightarrow \infty$ ([**R2**], Theorem 2.17). Then

$$-\int_{O_n \setminus E} f \leq \int_E f = \int_{O_n} f - \int_{O_n \setminus E} f \leq m(O_n) - \int_{O_n \setminus E} f. \tag{5}$$

Since $f \in L^1[0, 1]$ and $m(O_n \setminus E) \rightarrow 0$, one has $\int_{O_n \setminus E} f \rightarrow 0$ as $n \rightarrow \infty$. Hence we obtain (4) by letting $n \rightarrow \infty$ in (5). Since the integrals of f and $1 - f$ over every measurable set $E \subset [0, 1]$ are positive, it follows easily that $f \geq 0$ and $1 - f \geq 0$ almost everywhere.

Second solution. We can use Lebesgue's Differentiation Theorem ([R2], Theorem 7.10): If $f \in L^1[0, 1]$, then for almost every point $0 < x < 1$,

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(t) dt = f(x).$$

Clearly the limit on the left is between 0 and 1 for almost every x .

B3. Quite informally, f_k is a positive bell-shaped function with a maximum value 1 at $x = p_k/q_k$, and as $k \rightarrow \infty$ the graph of f_k gets more and more concentrated near p_k/q_k .

We must show that given $\varepsilon > 0$ there exists an N such that for all $k \geq N$,

$$m\{x \in [0, 1] : |f_k(x)| > \varepsilon\} < \varepsilon.$$

A brief computation using the formula for f_k shows that

$$m\{x \in [0, 1] : |f_k(x)| > \varepsilon\} = \frac{2\sqrt{-\log \varepsilon}}{q_k}.$$

Since $q_k \rightarrow \infty$ and ε is fixed, $2\sqrt{-\log \varepsilon}/q_k < \varepsilon$ for large k . Hence $f_k \rightarrow 0$ in measure.

Now we prove that $f_k(x)$ does not have a limit at any $x \in [0, 1]$. From the description of the f_k it follows that given x there exists a sequence $k_n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} f_{k_n}(x) = 0$. In fact, it suffices to choose k_n such that p_{k_n}/q_{k_n} stays outside a fixed neighborhood of x . On the other hand, given any integer $q > 0$, there exists an integer $0 \leq p \leq q$ such that $|x - p/q| \leq 1/q$, or $|p - qx| \leq 1$. Applying this for $q = 1, 2, 3, \dots$, and calling the resulting rational numbers r_{i_q} , it follows that $\liminf_{q \rightarrow \infty} f_{i_q}(x) \geq e^{-1} > 0$.

B4. Let $f_n(x) = x_n$ (the “ n -th digit function”). Evidently each f_n is measurable since it is a step function. It follows that $f = \sup_n f_n$ is also measurable.

We claim that $f(x) = 9$ for almost every x . The proof of this fact is identical to problem A6: Let A_n be the set of points $x = 0.x_1x_2x_3 \dots$ with $x_j = 9$ for some $1 \leq j \leq n$. Then it is easy to see that $m(A_n) = \sum_{j=1}^n 9^{j-1}10^{-j}$. Hence the set of all x with a digit 9 somewhere has measure $\lim_{n \rightarrow \infty} m(A_n) = \sum_{j=1}^{\infty} 9^{-1}0.9^j = 1$.

B5. Clearly E is a subset of the rectangle $[-\pi/6, \pi/6] \times [-1, 1]$. Let

$$F := ([-\pi/6, \pi/6] \times [-1, 1]) \setminus E.$$

Then F consists of a countable collection of segments of slope -1 corresponding to the lines $x + y \in \arccos(\mathbb{Q})$. Hence F has measure zero, which implies $m(E) = 2\pi/3$.

B6. The answer is negative. Let $\varphi : [0, 1] \rightarrow [0, 1]$ be the Cantor function (also known as the “devil’s staircase;” see e.g. [R2], Section 7.16) and extend it continuously to the real line by setting $\varphi(x) = 0$ for $x < 0$ and $\varphi(x) = 1$ for $x > 0$. To obtain a strictly increasing function, we modify φ by setting $\psi(x) := \varphi(x) + x$. The function ψ is a homeomorphism of the real line which fails to be absolutely continuous since φ fails to be so. Therefore there exists a set C of measure zero whose image $E := \psi(C)$ has positive measure. Every set of positive measure has non-measurable subsets ([R2], Theorem 2.22). Let $A \subset E$ be non-measurable. Note that $\chi_A \circ \psi = \chi_{\psi^{-1}(A)}$ is measurable since $\psi^{-1}(A) \subset C$, being a subset of a set of measure zero, is measurable. Now set $f := \psi^{-1}$ and $g := \chi_{\psi^{-1}(A)}$. Then $g \circ f = \chi_A$ is non-measurable.

B7. Fix $k \geq 1$ and divide $[0, 1]$ into 2^k dyadic intervals

$$J_i^k := \left[\frac{i-1}{2^k}, \frac{i}{2^k} \right) \quad 1 \leq i \leq 2^k - 1, \quad J_{2^k}^k := \left[\frac{2^k - 1}{2^k}, 1 \right].$$

Define $n_k(y)$ as the number of intervals J_i^k which contain at least one solution of $f(x) = y$. Evidently

$$n_1(y) \leq n_2(y) \leq \cdots \leq n_k(y) \leq \cdots \leq n(y).$$

If $n(y) = n < +\infty$ and $f(x_j) = y$ for $1 \leq j \leq n$, then for large k the points x_j will belong to distinct intervals J_i^k . Hence $n_k(y) = n$ for all large k . On the other hand, if $n(y) = +\infty$, then for any $N > 0$ choose N preimages of y and apply the above argument. It follows that $n_k(y) \geq N$ for all large k , which means $n_k(y) \rightarrow +\infty$ as $k \rightarrow \infty$. Thus, in any case,

$$n(y) = \lim_{k \rightarrow \infty} n_k(y).$$

So $y \mapsto n(y)$ is measurable as soon as we check each $y \mapsto n_k(y)$ is.

Let $m := \inf_{x \in [0,1]} f(x)$ and $M := \sup_{x \in [0,1]} f(x)$, and for $k \geq 1$ let

$$m_i^k := \inf_{x \in J_i^k} f(x) \quad M_i^k := \sup_{x \in J_i^k} f(x) \quad (1 \leq i \leq 2^k).$$

Clearly $m \leq m_i^k \leq M_i^k \leq M$. For every $m < y < M$ which is not of the form m_i^k or M_i^k , the function $y \mapsto n_k(y)$ is locally constant near y by the Intermediate Value Theorem. Hence $y \mapsto n_k(y)$ will be measurable.

B8. Fix $\nu > 2$ and let

$$E_{n,q} := \left\{ x \in [0, 1] : \left| x - \frac{p}{q} \right| < \frac{1}{nq^\nu} \text{ for some } 0 \leq p \leq q \right\}.$$

It is easy to see that

$$[0, 1] \setminus D_\nu = \bigcap_{n=1}^{\infty} \bigcup_{q=1}^{\infty} E_{n,q}.$$

Each $E_{n,q}$ is a disjoint union of $q - 1$ open intervals each of length $2/(nq^\nu)$ and two half-open intervals containing 0 and 1, each of length $1/(nq^\nu)$. It follows that

$$m(E_{n,q}) = q \cdot \frac{2}{nq^\nu} = \frac{2}{n} q^{-\nu+1},$$

which implies

$$m\left(\bigcup_{q=1}^{\infty} E_{n,q}\right) \leq \frac{2}{n} \sum_{q=1}^{\infty} q^{-\nu+1} =: \frac{C(\nu)}{n}.$$

Hence

$$m([0, 1] \setminus D_\nu) = \lim_{n \rightarrow \infty} m\left(\bigcup_{q=1}^{\infty} E_{n,q}\right) \leq \lim_{n \rightarrow \infty} \frac{C(\nu)}{n} = 0.$$

B9. By looking at the decimal expansion of x , it is not hard to see that

$$x \in E_n \iff 10^{n-1}x - [10^{n-1}x] < 10^{-\log n}.$$

By examining the graph of the piecewise linear function $x \mapsto 10^{n-1}x - [10^{n-1}x]$ on $[0, 1]$ we see that $m(E_n) = 10^{-\log n}$. Hence $\sum_{n=1}^{\infty} m(E_n) = \sum_{n=1}^{\infty} 10^{-\log n} = \sum_{n=1}^{\infty} n^{-\log 10} < \infty$. By Borel-Cantelli ([**R2**], Theorem 1.41), we conclude that $m(E) = 0$.

B10. We prove that $m(F) = 0$. Assume by way of contradiction that $m(F) > 0$. Since $m(E) > 0$ by the assumption, we can choose density points $a \in F$ and $b \in E$ and open balls $B := B(a, r)$ and $B' := B(b, r)$ (of equal radii) such that

$$m(B \cap F) \geq \frac{2}{3}m(B) \quad \text{and} \quad m(B' \cap E) \geq \frac{2}{3}m(B') = \frac{2}{3}m(B).$$

Now choose a sequence $x_k \in \mathbb{Q}^n$ such that $x_k + b \rightarrow a$ as $k \rightarrow \infty$, and set $B'_k := x_k + B'$. Then

$$m(B'_k \cap (x_k + E)) = m(B' \cap E) \geq \frac{2}{3}m(B)$$

and

$$B'_k \cap (x_k + E) \cap F = \emptyset.$$

Hence

$$m(B'_k \cap F) \leq \frac{1}{3}m(B).$$

Since $B'_k \rightarrow B$ as $k \rightarrow \infty$, we obtain $m(B \cap F) \leq (1/3)m(B)$, which is a contradiction.

B11. First assume that K is a convex finite-sided polygon. Then K_ε is the union of K together with a finite number of rectangles of height ε leaning on each side of K , together with finitely many circular wedges of radius ε centered at each vertex of K . It easily follows that

$$\text{area}(K_\varepsilon) = \text{area}(K) + \text{length}(\partial K)\varepsilon + \pi\varepsilon^2. \quad (6)$$

In general, for an arbitrary convex K , take a sequence of convex finite-sided polygons K^n approximating K uniformly, and note that $\text{area}(K^n) \rightarrow \text{area}(K)$, $\text{length}(\partial K^n) \rightarrow \text{length}(\partial K)$ and for a fixed $\varepsilon > 0$, $\text{area}(K_\varepsilon^n) \rightarrow \text{area}(K_\varepsilon)$. Hence (6) holds for general K as well.

It is interesting to note that the discriminant

$$(\text{length}(\partial K))^2 - 4\pi \text{area}(K)$$

of the quadratic expression in (6) is always non-negative by the isoperimetric inequality; it is zero precisely when K is a closed disk.

B12. All the properties of a metric are immediate except for the Triangle Inequality. For the latter, it suffices to note that

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

for all $a, b \in \mathbb{C}$. Setting $a := f - h$ and $b := h - g$, it follows that

$$\rho(f - g) = \int_X \frac{|f - g|}{1 + |f - g|} \leq \int_X \frac{|f - h|}{1 + |f - h|} + \int_X \frac{|h - g|}{1 + |h - g|} = \rho(f - h) + \rho(h - g),$$

proving that ρ is a metric on \mathcal{M} .

Now let us prove the second claim. First assume that $f_n \rightarrow f$ in measure and fix $\varepsilon > 0$. Then there exists an $N > 0$ such that $\mu(X_n(\varepsilon)) < \varepsilon$ for $n > N$, where

$$X_n(\varepsilon) := \{x \in X : |f_n(x) - f(x)| > \varepsilon\}.$$

Thus for $n > N$,

$$\begin{aligned} \rho(f_n - f) &= \left(\int_{X_n(\varepsilon)} + \int_{X \setminus X_n(\varepsilon)} \right) \frac{|f_n - f|}{1 + |f_n - f|} d\mu \\ &\leq \mu(X_n(\varepsilon)) + \int_{X \setminus X_n(\varepsilon)} \frac{\varepsilon}{1 + \varepsilon} d\mu \\ &\leq \varepsilon + \mu(X) \frac{\varepsilon}{1 + \varepsilon}. \end{aligned}$$

Hence $\rho(f_n - f) \rightarrow 0$ as $n \rightarrow \infty$.

Conversely, let $\rho(f_n - f) \rightarrow 0$, fix $\varepsilon > 0$ and choose $N > 0$ so large that $\rho(f_n - f) < \varepsilon^2/(1 + \varepsilon)$ for all $n > N$. Then, for any such n ,

$$\mu(X_n(\varepsilon)) \frac{\varepsilon}{1 + \varepsilon} \leq \int_{X_n(\varepsilon)} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \leq \int_X \frac{|f_n - f|}{1 + |f_n - f|} d\mu < \frac{\varepsilon^2}{1 + \varepsilon},$$

which implies $\mu(X_n(\varepsilon)) < \varepsilon$. Hence $f_n \rightarrow f$ in measure.

B13. Note that

$$G(x) = \frac{1}{x} - k \quad \text{if } \frac{1}{k+1} < x \leq \frac{1}{k}.$$

First we prove that $\mu(a, b) = \mu(G^{-1}(a, b))$ for any open interval $(a, b) \subset [0, 1]$. Since

$$G^{-1}(a, b) = \bigcup_{k=1}^{\infty} \left(\frac{1}{b+k}, \frac{1}{a+k} \right),$$

we have

$$\begin{aligned} \mu(G^{-1}(a, b)) &= \sum_{k=1}^{\infty} \mu \left(\frac{1}{b+k}, \frac{1}{a+k} \right) \\ &= \sum_{k=1}^{\infty} \frac{1}{\log 2} \int_{\frac{1}{b+k}}^{\frac{1}{a+k}} \frac{1}{1+x} dx \\ &= \sum_{k=1}^{\infty} \frac{1}{\log 2} \left[\log \left(\frac{a+k+1}{a+k} \right) - \log \left(\frac{b+k+1}{b+k} \right) \right] \\ &= \frac{1}{\log 2} \sum_{k=1}^{\infty} [\log(a+k+1) - \log(a+k)] - [\log(b+k+1) - \log(b+k)] \\ &= \frac{1}{\log 2} (-\log(a+1) + \log(b+1)) \\ &= \frac{1}{\log 2} \log \left(\frac{b+1}{a+1} \right) \\ &= \mu(a, b). \end{aligned}$$

Since every open set in $[0, 1]$ is a countable union of disjoint open intervals, it follows that $\mu(E) = \mu(G^{-1}(E))$ for every open set E . In general, let E be an arbitrary measurable set and take a decreasing sequence $\{U_n\}$ of open sets containing E such that $m(U_n \setminus E) \rightarrow 0$. Since μ is absolutely continuous with respect to m , it follows that $\mu(U_n \setminus E) \rightarrow 0$ as well. Clearly $\bigcap_{n=1}^{\infty} U_n = E \cup N$, where $m(N) = 0$. In particular, $G^{-1}(E) \cup G^{-1}(N) = \bigcap_{n=1}^{\infty} G^{-1}(U_n)$. Note that $m(N) = 0$ implies $m(G^{-1}(N)) = 0$

since G is piecewise smooth. Hence $\mu(G^{-1}(N)) = 0$ as well. Finally,

$$\begin{aligned}\mu(E) &= \lim_{n \rightarrow \infty} \mu(U_n) = \lim_{n \rightarrow \infty} \mu(G^{-1}(U_n)) \\ &= \mu\left(\bigcap_{n=1}^{\infty} G^{-1}(U_n)\right) \\ &= \mu(G^{-1}(E) \cup G^{-1}(N)) = \mu(G^{-1}(E)).\end{aligned}$$

B14. $L^\infty[0, 1]$ is not separable by the following argument. Consider the family $\{f_t\}_{0 \leq t \leq 1}$ of L^∞ functions on $[0, 1]$ defined by $f_t := \chi_{[0,t]}$. Note that $\|f_t - f_s\|_\infty = 1$ if $t \neq s$. Now let $\{g_n\}_{n=1}^\infty$ be a dense sequence in $L^\infty[0, 1]$. To each f_t assign a $g_n = g_{n(t)}$ with $\|f_t - g_n\|_\infty < 1/2$. The assignment $t \mapsto n(t)$ is one-to-one, since if $n(t) = n(s)$, then

$$\|f_t - f_s\|_\infty \leq \|f_t - g_{n(t)}\|_\infty + \|g_{n(s)} - f_s\|_\infty < 1/2 + 1/2 = 1,$$

which implies $t = s$. This means that the interval $[0, 1]$ can be injected into \mathbb{N} , which is clearly impossible. Therefore, no sequence in $L^\infty[0, 1]$ can be dense.

B15. Let us prove the first equality. It follows from the definition of p_n that $p_n \leq (\|f\|_\infty)^n \mu(X)$ or $\sqrt[n]{p_n} \leq \|f\|_\infty \sqrt[n]{\mu(X)}$. Therefore

$$\limsup_{n \rightarrow \infty} \sqrt[n]{p_n} \leq \|f\|_\infty. \quad (7)$$

Next, given $\varepsilon > 0$ let $X(\varepsilon) := \{x \in X : |f(x)| > \|f\|_\infty - \varepsilon\}$ which has positive measure. Then

$$\sqrt[n]{p_n} \geq \left(\int_{X(\varepsilon)} |f|^n d\mu \right)^{1/n} \geq (\|f\|_\infty - \varepsilon) \sqrt[n]{\mu(X(\varepsilon))},$$

From which it follows that $\liminf_{n \rightarrow \infty} \sqrt[n]{p_n} \geq \|f\|_\infty - \varepsilon$. Letting $\varepsilon \rightarrow 0$, we obtain

$$\liminf_{n \rightarrow \infty} \sqrt[n]{p_n} \geq \|f\|_\infty. \quad (8)$$

Inequalities (7) and (8) together prove the first required equality.

Now we prove the second equality. Clearly $p_{n+1} \leq \|f\|_\infty p_n$, which shows that

$$\limsup_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} \leq \|f\|_\infty. \quad (9)$$

By Hölder Inequality (with $p = (n+1)/n$ and $q = n+1$), we have

$$p_n \leq \left(\int_X |f|^{n+1} d\mu \right)^{\frac{n}{n+1}} \left(\int_X d\mu \right)^{\frac{1}{n+1}}.$$

Hence $p_n^{n+1} \leq p_{n+1}^n \mu(X)$ or $p_{n+1}/p_n \geq \sqrt[n]{p_n/\mu(X)}$. Hence

$$\liminf_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} \geq \liminf_{n \rightarrow \infty} \sqrt[n]{p_n} = \|f\|_\infty \quad (10)$$

by the first equality. Now the second equality follows from (9) and (10).

B16. Fix a $\delta > 0$ and choose a continuous function g with compact support such that $\|f - g\|_p < \delta$ ([**R2**], Theorem 3.14). Let $[-R, R]$ contain the support of g . Since Lebesgue measure on \mathbb{R} is translation-invariant, we have

$$\begin{aligned} \|f - f_\varepsilon\|_p &\leq \|f - g\|_p + \|g - g_\varepsilon\|_p + \|g_\varepsilon - f_\varepsilon\|_p \\ &\leq 2\delta + \|g - g_\varepsilon\|_p. \end{aligned}$$

So it suffices to prove that $\|g - g_\varepsilon\|_p < \delta$ for all sufficiently small ε . Since g is uniformly continuous on \mathbb{R} , $\sup_{x \in \mathbb{R}} |g(x) - g_\varepsilon(x)| < \delta / (\sqrt[p]{4R})$ if ε is small. Therefore

$$\|g - g_\varepsilon\|_p = \left(\int_{\mathbb{R}} |g(x) - g_\varepsilon(x)|^p dx \right)^{\frac{1}{p}} \leq \left(\frac{4R\delta^p}{4R} \right)^{\frac{1}{p}} = \delta.$$

B17. First assume that such a sequence $\{f_n\}$ exists. For any $x, y \in [0, 1]$,

$$|f(x) - f(y)| = \lim_{n \rightarrow \infty} |f_n(x) - f_n(y)| \leq \limsup_{n \rightarrow \infty} \int_x^y |f'_n(t)| dt \leq M|x - y|,$$

so f is M -Lipschitz.

Conversely, assume that f is M -Lipschitz. Then f is absolutely continuous on $[0, 1]$, so by the Fundamental Theorem of Calculus ([**R2**], Theorem 7.20) the derivative f' exists almost everywhere on $[0, 1]$, $f' \in L^1[0, 1]$, and $f(x) - f(y) = \int_x^y f'(t) dt$ for all $x, y \in [0, 1]$. Note that $|f'| \leq M$ since f is M -Lipschitz. By Lusin's Theorem ([**R2**], Theorem 2.24), there exists a sequence of continuous functions g_n which converges almost everywhere to f' and $|g_n(x)| \leq M$ for every $x \in [0, 1]$. Define

$$f_n(x) := f(0) + \int_0^x g_n(t) dt.$$

Clearly f_n is continuously differentiable and $|f'_n(x)| = |g_n(x)| \leq M$ for all x and all n . Finally,

$$|f_n(x) - f(x)| = \left| \int_0^x (g_n(t) - f'(t)) dt \right| \leq \int_0^x |g_n(t) - f'(t)| dt \rightarrow 0$$

by Lebesgue's Dominated Convergence Theorem.

B18. First, by Fatou's Lemma,

$$\int_X |f|^p d\mu = \int_X \liminf_{n \rightarrow \infty} |f_n|^p d\mu \leq \liminf_{n \rightarrow \infty} \int_X |f_n|^p d\mu < C,$$

which shows $f \in L^p(X, \mu)$. By Hölder,

$$\int_X |f| d\mu \leq \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}} \left(\int_X d\mu \right)^{\frac{1}{q}} = \|f\|_p \sqrt[q]{\mu(X)},$$

so $f \in L^1(X, \mu)$.

To prove the second assertion, let $\varepsilon > 0$. By Egoroff's Theorem ([**R2**], Exercise 3.16), there exists a measurable set $E \subset X$ with $\mu(X \setminus E) < \varepsilon$ such that f_n converges uniformly to f on E , so that

$$\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$$

if n is sufficiently large. It follows that

$$\begin{aligned} \|f_n - f\|_1 &= \left(\int_E + \int_{X \setminus E} \right) |f_n - f| d\mu \\ &\leq \varepsilon \mu(E) + \left(\int_{X \setminus E} |f_n - f|^p d\mu \right)^{\frac{1}{p}} \left(\int_{X \setminus E} d\mu \right)^{\frac{1}{q}} \\ &\leq \varepsilon \mu(X) + (\|f_n\|_p + \|f\|_p) \sqrt[q]{\mu(X \setminus E)} \\ &\leq \varepsilon \mu(X) + 2\sqrt[p]{C} \sqrt[q]{\varepsilon}. \end{aligned}$$

This proves $\|f_n - f\|_1 \rightarrow 0$ as $n \rightarrow \infty$.

B19. For $n \geq 1$, set

$$X_n := \{x \in X : 2^{-n}\|f\|_\infty < |f(x)| \leq 2^{-(n-1)}\|f\|_\infty\}.$$

Then the X_n are disjoint and $\mu(X_n) \leq C(2^{-n}\|f\|_\infty)^{-\alpha}$. Therefore

$$\int_X |f| d\mu = \sum_{n=1}^{\infty} \int_{X_n} |f| d\mu \leq \sum_{n=1}^{\infty} (2^{-(n-1)}\|f\|_\infty) C(2^{-n}\|f\|_\infty)^{-\alpha} = 2C\|f\|_\infty^{1-\alpha} \sum_{n=1}^{\infty} 2^{n\alpha-n}.$$

Since $0 < \alpha < 1$, the last series converges and we conclude that $f \in L^1(X, \mu)$.

B20. Define $g_n := f_n/\alpha_n$ and set $E_n := \{x \in X : |g_n(x)| > 1\}$ so that $\sum \mu(E_n) < \infty$. By Borel-Cantelli ([**R2**], Theorem 1.41), almost every $x \in X$ belongs to at most finitely many of the E_n . In other words, for almost every x there exists an $N = N(x) > 0$ such that $x \notin \bigcup_{n=N}^{\infty} E_n$. This means $|g_n(x)| \leq 1$ or $-1 \leq g_n(x) \leq 1$ if $n \geq N$. Hence

$$-1 \leq \liminf_{n \rightarrow \infty} g_n(x) \leq \limsup_{n \rightarrow \infty} g_n(x) \leq 1$$

which is what we wanted to prove.

B21. First solution. Let $x \in E$ be a density point. For every $0 < \varepsilon < 1/6$ there exists a $\delta > 0$ such that if $B = B(x, \delta)$ is the Euclidean δ -neighborhood of x , then $m(B \cap E) > (1 - \varepsilon)m(B)$. There exists a constant $\eta = \eta(\delta) > 0$ such that if $p \in \mathbb{R}^n$ and $|p| < \eta$, then the two balls B and $B + p$ intersect in an open set A with

$$\frac{2}{3}m(B) < m(A) < m(B).$$

We claim that every $p \in \mathbb{R}^n$ with $|p| < \eta$ belongs to $E - E$. In other words, we want to show that $E \cap E_p \neq \emptyset$, where $E_p := E + p$. To see this, first note that $m(A \cap E) > (1/2)m(A)$; otherwise

$$m(B \cap E) \leq m(A \cap E) + m(B \setminus A) \leq \frac{1}{2}m(A) + \frac{1}{3}m(B) \leq \frac{5}{6}m(B),$$

implying $1 - \varepsilon < 5/6$ which is false. Therefore, $m(A \cap E) > (1/2)m(A)$ and similarly $m(A \cap E_p) > (1/2)m(A)$. Now if $E \cap E_p = \emptyset$, then

$$m(A) \geq m((A \cap E) \cup (A \cap E_p)) = m(A \cap E) + m(A \cap E_p) > m(A),$$

which is absurd. So $E \cap E_p$ is non-empty, or $p \in E - E$.

Second solution. We prove that if $E, F \subset \mathbb{R}^n$ with $m(E) > 0$ and $m(F) > 0$, then $E + F$ contains an open ball. Without loss of generality we can assume that $m(E)$ and $m(F)$ are both finite. Consider the characteristic functions χ_E and χ_F which are both in $L^1(\mathbb{R}^n)$. First we note that the convolution

$$h(x) = (\chi_E * \chi_F)(x) := \int_{\mathbb{R}^n} \chi_E(x - t)\chi_F(t) dt$$

is continuous. This simply follows from the fact that for a function $f \in L^1(\mathbb{R}^n)$ and its translation $f_\varepsilon(x) := f(x + \varepsilon)$, the L^1 -norm $\|f - f_\varepsilon\|_1$ goes to 0 as $\varepsilon \rightarrow 0$ (see problem B16). Next we observe that h cannot be identically zero, since otherwise its Fourier transform

$$\hat{h}(s) := \int_{\mathbb{R}^n} h(x)e^{-ix \cdot s} dx = \hat{\chi}_E(s)\hat{\chi}_F(s)$$

would be identically zero. In particular, $m(E)m(F) = \hat{\chi}_E(0)\hat{\chi}_F(0) = 0$, which would contradict our assumption.

Now choose a ball $B \subset \mathbb{R}^n$ so that $h(x) \neq 0$ for $x \in B$. It follows that for each $x \in B$, the product $\chi_E(x - t)\chi_F(t)$ is not identically zero. Hence there exists a $t \in \mathbb{R}^n$ such that $t \in F$ and $x - t \in E$. This simply means that $x \in E + F$. Since x was arbitrary, we conclude that $B \subset E + F$.

B22. (a) For $n \geq 0$, set

$$X_n := \{x \in X : |f(x)| \geq 2^n\}.$$

First assume that $\sum_{n=1}^{\infty} 2^n \mu(X_n) < \infty$. In particular, $\mu(\bigcap_{n=1}^{\infty} X_n) = \lim_{n \rightarrow \infty} \mu(X_n) = 0$. Write

$$\begin{aligned} \int_X |f| d\mu &= \int_{\{x: |f(x)| < 2\}} |f| d\mu + \sum_{n=1}^{\infty} \int_{X_n \setminus X_{n+1}} |f| d\mu \\ &\leq 2\mu(X) + \sum_{n=1}^{\infty} 2^{n+1} \mu(X_n). \end{aligned} \tag{11}$$

The last series converges by the assumption, so $f \in L^1(X, \mu)$.

Now assume that $f \in L^1(X, \mu)$. Write S_N for the partial sum $\sum_{n=1}^N 2^n \mu(X_n)$. Then

$$\begin{aligned} S_N &= \sum_{n=1}^N 2^n (\mu(X_{n+1}) + \mu(X_n \setminus X_{n+1})) \\ &= \frac{1}{2} \sum_{n=1}^N 2^{n+1} \mu(X_{n+1}) + \sum_{n=1}^N 2^n \mu(X_n \setminus X_{n+1}) \\ &= \frac{1}{2} S_N + 2^N \mu(X_{N+1}) - \mu(X_1) + \sum_{n=1}^N 2^n \mu(X_n \setminus X_{n+1}) \end{aligned}$$

from which it follows that

$$S_N = 2 \sum_{n=1}^N 2^n \mu(X_n \setminus X_{n+1}) + 2^{N+1} \mu(X_{N+1}) - 2\mu(X_1). \tag{12}$$

By the first equality in (11) above, $\sum_{n=1}^N 2^n \mu(X_n \setminus X_{n+1}) \leq \|f\|_1$. Moreover, for every n

$$\|f\|_1 \geq \int_{X_n} |f| d\mu \geq 2^n \mu(X_n)$$

so that $2^{N+1} \mu(X_{N+1}) \leq \|f\|_1$. Hence by (12)

$$S_N \leq 2\|f\|_1 + \|f\|_1 - 2\mu(X_1)$$

which shows S_N is a bounded sequence. Hence $\lim_{N \rightarrow \infty} S_N$ is finite.

(b) The proof is quite similar to part (a). This time define

$$X_n := \{x \in X : |f(x)| \geq 2^{-n}\}.$$

First assume that $\sum_{n=1}^{\infty} 2^{-n} \mu(X_n) < \infty$. Note in particular that this implies $\mu(X_0) \leq \mu(X_1) < \infty$. Write

$$\begin{aligned} \int_X |f| d\mu &= \int_{X_0} |f| d\mu + \sum_{n=1}^{\infty} \int_{X_n \setminus X_{n-1}} |f| d\mu \\ &\leq \mu(X_0) \|f\|_{\infty} + \sum_{n=1}^{\infty} 2^{-(n-1)} \mu(X_n). \end{aligned} \quad (13)$$

The last series converges by the assumption, so $f \in L^1(X, \mu)$.

Now assume that $f \in L^1(X, \mu)$. Write S_N for the partial sum $\sum_{n=1}^N 2^{-n} \mu(X_n)$. Then

$$\begin{aligned} S_N &= \sum_{n=1}^N 2^{-n} (\mu(X_{n-1}) + \mu(X_n \setminus X_{n-1})) \\ &= \frac{1}{2} \sum_{n=1}^N 2^{-(n-1)} \mu(X_{n-1}) + \sum_{n=1}^N 2^{-n} \mu(X_n \setminus X_{n-1}) \\ &= \frac{1}{2} S_N - 2^{-(N+1)} \mu(X_N) + \frac{1}{2} \mu(X_0) + \sum_{n=1}^N 2^{-n} \mu(X_n \setminus X_{n-1}) \end{aligned}$$

from which it follows that

$$S_N = 2 \sum_{n=1}^N 2^{-n} \mu(X_n \setminus X_{n-1}) - 2^{-N} \mu(X_N) + \mu(X_0). \quad (14)$$

By the first equality in (13) above, $\sum_{n=1}^N 2^{-n} \mu(X_n \setminus X_{n-1}) \leq \|f\|_1$. Moreover, for every n

$$\|f\|_1 \geq \int_{X_0} |f| d\mu \geq \mu(X_0).$$

Hence by (14)

$$S_N \leq 2\|f\|_1 + \|f\|_1 - 2^{-N} \mu(X_N)$$

which shows S_N is a bounded sequence. Hence $\lim_{N \rightarrow \infty} S_N$ is finite.

B23. Let $\lambda_i \in \mathbb{R}$ be the eigenvalues of A . There exists an orthogonal transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$TAT^t = \text{diag}(\lambda_1, \dots, \lambda_n).$$

Let $Tx = y$ and define a new function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ by $g(y) := f(x) = f(T^t y)$. Note that

$$g(y) = e^{-(T^t y)^t A T^t y} = e^{-y^t T A T^t y},$$

or

$$g(y_1, \dots, y_n) = e^{-\sum \lambda_i y_i^2}.$$

Since $|\det T| = 1$, it follows from the change of variable formula that

$$\int_{\mathbb{R}^n} g(y) dy = \int_{\mathbb{R}^n} f(x) dx.$$

If A is positive-definite, then $\lambda_i > 0$ for all i and by the Fubini Theorem

$$\int_{\mathbb{R}^n} g(y) dy = \prod_{i=1}^n \int_{-\infty}^{\infty} e^{-\lambda_i y_i^2} dy_i = \prod_{i=1}^n \sqrt{\frac{\pi}{\lambda_i}} = \frac{\pi^{n/2}}{\sqrt{\det A}},$$

so $f \in L^1(\mathbb{R}^n)$.

Conversely, if $\lambda_i \leq 0$ for some i , then the corresponding factor $\int_{-\infty}^{\infty} e^{-\lambda_i y_i^2} dy_i$ diverges, hence $g \notin L^1(\mathbb{R}^n)$. Hence $f \notin L^1(\mathbb{R}^n)$ if A is not positive-definite.

B24. Let $f(x) := \sum_{n=1}^{\infty} |c_n f_n(x)|$, which is finite for almost every $x \in [0, 1]$. For $t > 0$, set $E := \{x \in [0, 1] : f(x) > t\}$. Since $f(x) < +\infty$ for almost every x , $m(E) \rightarrow 0$ as $t \rightarrow \infty$. Note that by Lebesgue's Monotone Convergence Theorem,

$$\sum_{n=1}^{\infty} |c_n| \int_{E^c} |f_n(x)| dx = \int_{E^c} f(x) dx \leq t m(E^c) \leq t. \quad (15)$$

On the other hand, by Cauchy-Schwarz

$$\begin{aligned} \int_{E^c} |f_n(x)| dx &= \int_0^1 |f_n(x)| dx - \int_E |f_n(x)| dx \\ &\geq A - \left(\int_E |f_n(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_E dx \right)^{\frac{1}{2}} \\ &\geq A - B \sqrt{m(E)}. \end{aligned} \quad (16)$$

If t is so large that $A - B \sqrt{m(E)} > 0$, we conclude from (15) and (16) that

$$\sum_{n=1}^{\infty} |c_n| (A - B \sqrt{m(E)}) \leq t,$$

or

$$\sum_{n=1}^{\infty} |c_n| \leq \frac{t}{A - B \sqrt{m(E)}} < +\infty.$$

B25. Λ is a positive linear functional on $C[0, 1]$ whose kernel contains S . By the Riesz Representation Theorem ([R2], Theorem 2.14), there exists a unique positive Borel measure μ which represents Λ :

$$\Lambda g = \int_0^1 g d\mu \text{ for all } g \in C[0, 1].$$

For $0 < t < 1$, take a non-negative continuous function g with $g(x) = 1$ if $x \geq t$ and $g(x) = 0$ if $x \leq t/2$ and interpolate linearly in between. Then $g \in S$ and

$$0 = \Lambda g = \int_0^1 g d\mu \geq \int_t^1 g d\mu = \mu[t, 1].$$

Since this is true for all $0 < t < 1$, we conclude that μ is a multiple of the unit mass concentrated at 0. Hence $\Lambda g = Mg(0)$, where $M := \Lambda 1 \geq 0$.

B26. The result is immediate for $p = +\infty$ (take $f \equiv 1$). So let us consider the case $1 \leq p < +\infty$. Assume by way of contradiction that $\varphi \notin L^\infty(\Omega)$. Then for $n \geq 1$, the set $E_n := \{x \in \Omega : |\varphi(x)| > n\}$ has positive measure. If $E := \bigcap E_n$ had positive measure, then we could choose a measurable set $F \subset E$ with $0 < m(F) < +\infty$ and $f = \chi_F$ would be in $L^p(\Omega)$ while $\varphi f \notin L^p(\Omega)$. Therefore $m(E) = 0$.

Now for each $n \geq 1$, consider $D_n := E_n \setminus E_{n+1}$. Note that there are infinitely many n for which $m(D_n) > 0$ (otherwise, φ would be essentially bounded). Arrange them in a sequence D_{n_1}, D_{n_2}, \dots and choose measurable sets $F_k \subset D_{n_k}$ such that $0 < d_k := m(F_k) < +\infty$. Let

$$c_k := \left(\frac{1}{kn_k^p d_k} \right)^{\frac{1}{p}},$$

and define $f : \Omega \rightarrow \mathbb{R}$ such that $f(x) = c_k$ if $x \in F_k$ ($k = 1, 2, \dots$) and $f(x) = 0$ otherwise. Then,

$$\int_{\Omega} |f|^p dm = \sum_{k=1}^{\infty} c_k^p d_k = \sum_{k=1}^{\infty} \frac{1}{kn_k^p} \leq \sum_{k=1}^{\infty} \frac{1}{k^{1+p}} < \infty,$$

while

$$\int_{\Omega} |\varphi f|^p dm \geq \sum_{k=1}^{\infty} n_k^p c_k^p d_k = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

This contradiction proves $\varphi \in L^\infty(\Omega)$.

B27. (a) For every integer $M \geq 1$, define

$$E_M := \{x \in X : |f_n(x)| \leq M \text{ for all } n\}.$$

Since $\{f_n(x)\}$ is a bounded sequence for each x , we have $X = \bigcup_{M \geq 1} E_M$. Each E_M is closed and X is complete, so by Baire's Category Theorem some E_M must have non-empty interior U . This proves (a).

(b) Fix $\varepsilon > 0$ and for every integer $N \geq 1$, define

$$F_N := \{x \in X : |f_n(x) - f_m(x)| \leq \varepsilon \text{ for all } n, m \geq N\}.$$

Since $\{f_n(x)\}$ converges for each x , we have $X = \bigcup_{N \geq 1} F_N$. Each F_N is closed and X is complete, so by Baire's Category Theorem some F_N must have non-empty interior U . Then for all $x \in U$, $|f_n(x) - f_m(x)| \leq \varepsilon$ provided that $n, m \geq N$. Letting $m \rightarrow \infty$, we obtain $|f_n(x) - f(x)| \leq \varepsilon$ for $n \geq N$, which proves (b).

C. Banach and Hilbert Spaces

C1. By the Riesz Representation Theorem, there exist $\alpha, \beta \in H$ such that

$$f(x) = \langle x, \alpha \rangle, \quad g(x) = \langle x, \beta \rangle$$

for all $x \in H$. Moreover, it is easy to see that $\|\alpha\| = \|f\| = \|g\| = \|\beta\|$. Now, $\ker(f) = \alpha^\perp \subset \ker(g) = \beta^\perp$. Taking orthogonal complements, we conclude that the subspace generated by β is contained in the subspace generated by α . Hence $\alpha = c\beta$ for some $c \in \mathbb{C}$. But $\|\alpha\| = \|\beta\|$ implies $|c| = 1$ so $c = e^{-ir}$ for some $r \in \mathbb{R}$. It follows that $f(x) = e^{ir}g(x)$.

C2. Choose a base point $b \in K$. For $0 < \lambda < 1$, define

$$T_\lambda x := \lambda T x + (1 - \lambda)b.$$

Note that T_λ maps K to K since K is convex. Also

$$\|T_\lambda x - T_\lambda y\| = \lambda \|T x - T y\| \leq \lambda \|x - y\|.$$

Since K is complete (as a closed subset of a Banach space), it follows from the Contraction Mapping Principle that T_λ has a unique fixed point $x_\lambda \in K$. Now

$$\|T x_\lambda - x_\lambda\| = \left\| \frac{T_\lambda x_\lambda - (1 - \lambda)b}{\lambda} - x_\lambda \right\| = \left\| \left(\frac{1}{\lambda} - 1\right)x_\lambda - \left(\frac{1}{\lambda} - 1\right)b \right\|,$$

which is bounded by $2(1/\lambda - 1) \text{diam } K$. Hence if λ is close enough to 1, $\|T x_\lambda - x_\lambda\|$ can be made arbitrarily small.

T does not necessarily have a fixed point, as can be seen from the following example: Let B be the Banach space of all sequences $x = \{x_n\}_{n \geq 1}$ of complex numbers with $\lim_{n \rightarrow \infty} x_n = 0$, equipped with the norm $\|x\| = \sup_n |x_n|$. Let K be the closed unit ball in B and define $T(\{x_n\})$, by

$$T(x_1) := \frac{1}{2}(1 + \|x\|), \quad T(x_n) := (1 - 2^{-n})x_{n-1} \text{ for } n \geq 2.$$

It is easy to verify that T maps K to itself, and satisfies $\|Tx - Ty\| < \|x - y\|$. But T has no fixed points, since $Tx = x$ would imply $|x_n| \geq \prod_{i=1}^n (1 - 2^{-i}) > \prod_{i=1}^{\infty} (1 - 2^{-i}) > 0$, contradicting $x_n \rightarrow 0$.

C3. Let $d > 0$ be the distance from 0 to the hyperplane $L := f^{-1}(1)$. Pick any $y \in L$ and note that $1 = |f(y)| \leq \|f\| \|y\|$, or $\|f\| \geq 1/\|y\|$. Taking the infimum over all $y \in L$, we obtain $\|f\| \geq 1/d$.

On the other hand, for every $\varepsilon > 0$, we can find an $x \in E$ such that $\|x\| = 1$ and $|f(x)| \geq \|f\| - \varepsilon$. Set $\alpha := f(x)$ and $y := x/\alpha$. Then $y \in L$ and so $d \leq \|y\| = 1/|\alpha|$, or $1/d \geq |f(x)| \geq \|f\| - \varepsilon$. Letting $\varepsilon \rightarrow 0$, we obtain $\|f\| \leq 1/d$.

C4. We prove that E is Banach. First observe that $\|x + F\| \leq \|x\|$ for all $x \in E$. Moreover, if $\|x_n + F\| \rightarrow 0$, then there exist $y_n \in x_n + F$ such that $\|y_n\| \rightarrow 0$.

Now assume $\{x_n\}$ is a Cauchy sequence in E . Then

$$\|(x_n + F) - (x_m + F)\| = \|(x_n - x_m) + F\| \leq \|x_n - x_m\| \rightarrow 0$$

as $n, m \rightarrow \infty$. Hence $\{x_n + F\}$ is a Cauchy sequence in E/F . So there exists an $a + F \in E/F$ such that $x_n + F \rightarrow a + F$ in E/F . This implies $\|(a + F) - (x_n + F)\| = \|(a - x_n) + F\| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, there exist $y_n \in (a - x_n) + F$ such that $\|y_n\| \rightarrow 0$. Put $z_n := a - y_n$. Then $z_n \rightarrow a$ in E , and $\{x_n - z_n\}$ is a Cauchy sequence in F . Hence $x_n - z_n \rightarrow w \in F$. It follows that $x_n \rightarrow w + a \in E$.

C5. By Closed Graph Theorem, it suffices to prove that if $x_n \in H$ converges to $x \in H$ and if Tx_n converges to $y \in H$, then $y = Tx$. Pick any $v \in H$ and observe that

$$\langle y, v \rangle = \lim_{n \rightarrow \infty} \langle Tx_n, v \rangle = \lim_{n \rightarrow \infty} \langle x_n, Tv \rangle = \langle x, Tv \rangle = \langle Tx, v \rangle.$$

In particular, for $v := Tx - y$, we obtain $\|Tx - y\|^2 = 0$, implying $Tx = y$.

To see the role of completeness, let H be the space of C^∞ functions $f : \mathbb{R} \rightarrow \mathbb{C}$ which vanish outside the unit interval, and equip H with the usual integral inner product. Define $T : H \rightarrow H$ by $(Tf)(x) := if'(x)$. Integration by Parts shows that T is self-adjoint. Evidently T fails to be continuous.

C6. The answer is negative. Let E be the space of continuous real-valued functions on the interval $[-1, 1]$, with the integral inner product, and let F be the closed subspace

of those functions $f \in E$ for which

$$\int_{-1}^0 f(x) dx = \int_0^1 f(x) dx.$$

Suppose that there exists a function $g \in E$ such that $\int_{-1}^1 g f = 0$ for all $f \in F$. Pick arbitrary $-1 < x < 0$ and $0 < y < 1$ and fix small disjoint intervals I centered at x and J centered at y with $m(I) = m(J)$. Choose a sequence of functions $f_n \in F$ approximating the characteristic function of $I \cup J$. Passing to the limit in $\int_{-1}^1 g f_n = 0$, we obtain

$$\frac{1}{m(I)} \int_I g(t) dt = -\frac{1}{m(J)} \int_J g(t) dt.$$

Since I and J can be made arbitrarily small and g is continuous, it follows that $g(x) = -g(y)$. Since this holds for all $-1 < x < 0$ and all $0 < y < 1$, it follows that for some constant C , $g(x) = C$ for $-1 < x < 0$ and $g(x) = -C$ for $0 < x < 1$. This is impossible unless $C = 0$.

C7. First note that the condition

$$\lim_{\|x\| \rightarrow \infty} \frac{|\langle Tx, x \rangle|}{\|x\|} = +\infty \quad (17)$$

implies

$$C := \inf_{\|x\|=1} |\langle Tx, x \rangle| > 0. \quad (18)$$

In fact, if (18) fails, there exist vectors x_n with $\|x_n\| = 1$ such that $\alpha_n := \langle Tx_n, x_n \rangle \rightarrow 0$. Choose a sequence of complex numbers λ_n going to infinity such that $\lambda_n \alpha_n \rightarrow 0$. Then

$$\frac{|\langle T(\lambda_n x_n), \lambda_n x_n \rangle|}{\|\lambda_n x_n\|} = \frac{|\lambda_n|^2 |\alpha_n|}{|\lambda_n|} \rightarrow 0$$

while $\|\lambda_n x_n\| \rightarrow \infty$. This contradicts (17).

Now (18) means

$$|\langle Tx, x \rangle| \geq C \|x\|^2 \quad (19)$$

for all $x \in H$. In particular, T is injective and $T : H \rightarrow T(H)$ is an isomorphism. To prove $T(H) = H$, first note that $T(H)$ is closed: If $Tx_n \rightarrow y \in H$ for a sequence $x_n \in H$, then by (19)

$$\|x_n - x_m\|^2 \leq \frac{1}{C} |\langle T(x_n - x_m), x_n - x_m \rangle| \leq \frac{1}{C} \|T(x_n - x_m)\| \|x_n - x_m\|$$

or

$$\|x_n - x_m\| \leq \frac{1}{C} \|T(x_n - x_m)\|,$$

implying that $\{x_n\}$ is a Cauchy sequence. Let $x_n \rightarrow x \in H$. Clearly $y = T(x)$ so $y \in T(H)$. Now choose any vector $v \in T(H)^\perp$. It follows that $\langle Tv, v \rangle = 0$, hence by (19), $v = 0$. So we must have $T(H) = H$.

C8. Define

$$\|x\|_T := \sup_{n \in \mathbb{Z}} \|T^n x\|.$$

It is easy to see that $\|\cdot\|_T$ is a norm, and that

$$\|x\| \leq \|x\|_T \leq C\|x\|$$

for all $x \in E$, implying that the two norms are equivalent. Finally, note that

$$\|Tx\|_T = \sup_{n \in \mathbb{Z}} \|T^{n+1}x\| = \sup_{n \in \mathbb{Z}} \|T^n x\| = \|x\|_T.$$

C9. (a) Let us first prove that each T_n is monic. Set $\alpha := \cos^{-1} x$ and write

$$\begin{aligned} \cos n\alpha + i \sin n\alpha &= (\cos \alpha + i \sin \alpha)^n \\ &= \sum_{j=0}^n \binom{n}{j} i^j \cos^{n-j} \alpha \sin^j \alpha \end{aligned}$$

from which it follows that

$$T_n(x) = \frac{1}{2^{n-1}} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (-1)^j x^{n-2j} (1-x^2)^j.$$

Therefore, the coefficient of x^n in $T_n(x)$ is

$$\frac{1}{2^{n-1}} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} = 1,$$

proving that T_n is monic.

Clearly $\sup_{-1 \leq x \leq 1} |T_n(x)| = 1/2^{n-1}$, taken at points where $\cos(n \cos^{-1} x) = \pm 1$, or $n \cos^{-1} x = k\pi$, or $x = \cos(k\pi/n)$, $k = 0, 1, \dots, n$.

(b) Let d be the required distance. Observe that

$$d = \inf \left\{ \sup_{-1 \leq x \leq 1} |P(x)| : P \text{ is a monic polynomial of degree } n \right\}.$$

By considering T_n , we see that $d \leq 1/2^{n-1}$. If $d < 1/2^{n-1}$, there exists a monic polynomial Q of degree n such that $\sup_{-1 \leq x \leq 1} |Q(x)| < 1/2^{n-1} = \sup_{-1 \leq x \leq 1} |T_n(x)|$. Then $Q - T_n$ will be a polynomial of degree $< n$ with at least n zeros since by (a) the graph of Q must intersect that of T_n in at least n points. This contradiction proves that $d = 1/2^{n-1}$.

C10. Note that by the Riesz Representation Theorem, weak convergence for a sequence $\{A_n\}$ of operators means

$$\langle A_n x, y \rangle \rightarrow \langle Ax, y \rangle$$

for all $x, y \in H$. Write

$$\begin{aligned} \|A_n x - Ax\|^2 &= \langle A_n x - Ax, A_n x - Ax \rangle \\ &= \|A_n x\|^2 + \|Ax\|^2 - 2 \operatorname{Re} \langle A_n x, Ax \rangle \\ &\rightarrow 2\|Ax\|^2 - 2 \operatorname{Re} \langle Ax, Ax \rangle = 0. \end{aligned}$$

This implies $A_n x \rightarrow Ax$ for all $x \in H$.

If A as well as all the A_n are unitary, then $A_n^* A_n = A_n A_n^* = I = A^* A = A A^*$. Hence both $\|A_n x\|$ and $\|Ax\|$ are equal to $\|x\|$. So the condition $\|A_n x\| \rightarrow \|Ax\|$ is automatically satisfied.

C11. The answer is negative. Assuming the existence of such a norm, consider the sequence $f_n \in C[0, 1]$ defined by

$$f(x) = \begin{cases} nx & 0 \leq x \leq \frac{1}{n} \\ -nx + 2 & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \leq 1 \end{cases}$$

which satisfies $f_n(x) \rightarrow 0$ for every $x \in [0, 1]$. Note that $\|f_n\| > 0$ since $f_n \not\equiv 0$. But then $g_n := f_n / \|f_n\|$ is continuous and has norm 1, while $g_n(x) \rightarrow 0$ for every $x \in [0, 1]$.

D. Fourier Series and Integrals

D1. Let

$$f(x) \sim \sum_{-\infty}^{\infty} \hat{f}(n) e^{inx}$$

be the Fourier series of f , where as usual

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$$

denotes the n -th Fourier coefficient of f . Since f' is continuous, Integration by Parts shows that it has Fourier series of the form

$$f'(x) \sim \sum_{-\infty}^{\infty} in\hat{f}(n)e^{inx}.$$

The condition $\int_0^{2\pi} f = 0$ implies $\hat{f}(0) = 0$. On the other hand, the condition $\int_0^{2\pi} (f + f')(f - f') = 0$ can be written as $\int_0^{2\pi} f^2 = \int_0^{2\pi} (f')^2$ which, by Parseval, translates into

$$\sum_{-\infty}^{\infty} |\hat{f}(n)|^2 = \sum_{-\infty}^{\infty} n^2 |\hat{f}(n)|^2,$$

or

$$\sum_{-\infty}^{\infty} (1 - n^2) |\hat{f}(n)|^2 = 0.$$

This can happen only if $\hat{f}(n) = 0$ for $n \neq 0, \pm 1$. It follows that f is a real linear combination of e^{ix} and e^{-ix} . Hence

$$f(x) = A \cos x + B \sin x$$

for some real constants A, B .

D2. Fix an $s \neq 0$ such that $|\hat{f}(s)| \geq \hat{f}(0)$. Then

$$\frac{1}{\sqrt{2\pi}} \left| \int_{-\infty}^{\infty} f(x)e^{-isx} dx \right| \geq \hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)e^{-isx}| dx.$$

Since equality occurs in Triangle Inequality, there is a constant $c \neq 0$ such that

$$f(x)e^{-isx} = c |f(x)e^{-isx}|$$

or

$$f(x)e^{-isx} = c f(x)$$

for almost every $x \in \mathbb{R}$. If E denotes the set of x for which $f(x) \neq 0$, then $e^{-isx} = c$ for every $x \in E$. But since $s \neq 0$ is fixed, this last equation has only countably many solutions in x , implying $m(E) = 0$.

D3. Let $f(t) = (x(t), y(t))$, where x and y are smooth 2π -periodic real-valued functions. Expand x and y in their Fourier series:

$$x(t) \sim \sum_{-\infty}^{\infty} \hat{x}(n)e^{int}, \quad y(t) \sim \sum_{-\infty}^{\infty} \hat{y}(n)e^{int}.$$

Without losing generality, we may assume that $\|f'\| \equiv 1$, or $(x')^2 + (y')^2 \equiv 1$ so that $L = 2\pi$. By Green's Theorem

$$A = \int_{\gamma} x dy \leq \frac{1}{2} \int_0^{2\pi} [x^2(t) + (y')^2(t)] dt = \frac{1}{2} \int_0^{2\pi} [x^2(t) + 1 - (x')^2(t)] dt.$$

It follows from Parseval's Identity that

$$A \leq \pi \sum_{-\infty}^{\infty} |\hat{x}(n)|^2 + \pi - \pi \sum_{-\infty}^{\infty} n^2 |\hat{x}(n)|^2$$

or

$$A \leq \pi + \pi |\hat{x}(0)|^2 + \pi \sum_{|n|>1} (1 - n^2) |\hat{x}(n)|^2 \leq \pi + \pi |\hat{x}(0)|^2.$$

By translating γ horizontally, we may assume that $\hat{x}(0) = \int_0^{2\pi} x(t) dt = 0$. Hence $A \leq \pi = L^2/(4\pi)$.

Equality holds if and only if $x(t) = y'(t)$ for all t and $\hat{x}(n) = 0$ for $|n| > 1$. It is easy to see that these occur if and only if γ is a circle.

D4. Consider the characteristic function χ_E and note that E is invariant under the rotation R_{θ} if and only if $\chi_E = \chi_E \circ R_{\theta}$. Expand both functions in their Fourier series:

$$\chi_E(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, \quad (\chi_E \circ R_{\theta})(x) \sim \sum_{-\infty}^{\infty} c_n e^{in(x+2\pi\theta)}.$$

It follows that

$$c_n = c_n e^{2\pi in\theta}$$

for all n . Since θ is irrational, this implies $c_n = 0$ if $n \neq 0$. Hence χ_E is constant almost everywhere, which means $\chi_E = 0$ or 1 almost everywhere.

If θ is rational, this result does not hold anymore. In fact, let $\theta = p/q$ with $q > 0$ and p and q be relatively prime. let $S \subset [0, 2\pi/q]$ be any measurable set such that $0 < m(S) < 2\pi/q$. Then

$$E := S \cup R_{p/q}(S) \cup R_{2p/q}(S) \cup \cdots \cup R_{(q-1)p/q}(S)$$

is invariant under $R_{p/q}$ and $0 < m(E) < 2\pi$.

D5. We have

$$\hat{f}(n) = -\frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-in(x-\frac{\pi}{n})} dx = -\frac{1}{2\pi} \int_0^{2\pi} f(x + \frac{\pi}{n}) e^{-inx} dx.$$

Hence

$$2\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} \left(f(x) - f\left(x + \frac{\pi}{n}\right) \right) e^{-inx} dx.$$

It follows that

$$|\hat{f}(n)| \leq \frac{1}{4\pi} \cdot 2\pi \cdot M \left(\frac{\pi}{|n|} \right)^\alpha$$

or

$$|\hat{f}(n)| \leq \frac{M\pi^\alpha}{2} \frac{1}{|n|^\alpha}.$$

D6. Since f is of bounded variation, it is differentiable almost everywhere and its derivative f' is integrable. Integration by Parts then shows that

$$\hat{f}(n) = \frac{1}{2\pi in} \int_0^{2\pi} f'(x) e^{-inx} dx.$$

By Riemann-Lebesgue Lemma ([R2], Section 5.14) applied to f' , the last integral tends to 0 as $|n|$ tends to infinity. Hence $|n\hat{f}(n)| \rightarrow 0$ as $|n| \rightarrow \infty$.

D7. Fix $f \in H$. Since the Fourier series of f converges to f in $L^2(\mathbb{T})$, there exists an increasing sequence of integers $\{n_k\}$ such that

$$\lim_{k \rightarrow \infty} \sum_{n=-n_k}^{n_k} \hat{f}(n) e^{inx} = f(x)$$

for almost every x . It follows from Cauchy-Schwarz that for every such x ,

$$|f(x)| \leq \lim_{k \rightarrow \infty} \sum_{n=-n_k}^{n_k} |\hat{f}(n)| = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq \left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 (1+n^2) \right)^{\frac{1}{2}} \left(\sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \right)^{\frac{1}{2}}.$$

Setting $C := (\sum_{n=-\infty}^{\infty} 1/(1+n^2))^{1/2}$, we obtain $|f(x)| \leq C\|f\|_H$ for almost every x .

D8. *First solution.* Expand f into its Fourier series

$$f(x, y) \sim \sum_{m, n \in \mathbb{Z}} c_{mn} e^{2\pi imx} e^{2\pi iny}$$

and note that the Laplacian of f corresponds to the Fourier series

$$\Delta f(x, y) \sim -4\pi^2 \sum_{m, n \in \mathbb{Z}} (m^2 + n^2) c_{mn} e^{2\pi imx} e^{2\pi iny}.$$

If $\Delta f = \lambda f$ for some $\lambda \in \mathbb{R}$, it follows that

$$[\lambda + 4\pi^2(m^2 + n^2)]c_{mn} = 0$$

for all m, n . Since $f \not\equiv 0$, there exists some pair (m, n) for which $c_{mn} \neq 0$. Therefore $\lambda = -4\pi^2(m^2 + n^2) \leq 0$.

Second solution. Suppose that $\Delta f = \lambda f$ and set $g := f^2 \geq 0$. Then

$$\Delta g = 2(f_x^2 + f_y^2) + 2f \Delta f = 2(f_x^2 + f_y^2) + 2\lambda g.$$

Let S denote the unit square in \mathbb{R}^2 and note that by Green's Theorem

$$\int \int_S \Delta g = \int_{\partial S} -g_y dx + g_x dy = 0$$

since the partial derivatives of g are doubly periodic. It follows that

$$\int \int_S (f_x^2 + f_y^2) + \lambda \int \int_S g = 0$$

from which we conclude that $\lambda \leq 0$.

D9. Note that if f has Fourier series $f(x) \sim \sum_{-\infty}^{\infty} \hat{f}(n)e^{inx}$, then the Fourier series of f_ε is of the form

$$f_\varepsilon(x) \sim \sum_{-\infty}^{\infty} \hat{f}(n)e^{in\varepsilon}e^{inx}.$$

By Parseval's Identity, we have

$$\|f - f_\varepsilon\|_2^2 = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2 |1 - e^{in\varepsilon}|^2.$$

If f fails to be constant almost everywhere, there exists an $n \neq 0$ for which $\hat{f}(n) \neq 0$. It follows that

$$\|f - f_\varepsilon\|_2 \geq |\hat{f}(n)| |1 - e^{in\varepsilon}| = 2|\hat{f}(n)| \left| \sin\left(\frac{n\varepsilon}{2}\right) \right|,$$

which implies

$$\liminf_{\varepsilon \rightarrow 0} \frac{\|f - f_\varepsilon\|_2}{|\varepsilon|} \geq \liminf_{\varepsilon \rightarrow 0} 2|\hat{f}(n)| \frac{|\sin(n\varepsilon/2)|}{|\varepsilon|} = |n\hat{f}(n)| > 0.$$

D10. *First solution.* Recall the proof of the classical result that trigonometric polynomials are dense in $C(\mathbb{T})$ ([**R2**], Theorem 4.25): We first construct a sequence of trigonometric polynomials Q_n converging weakly to the unit mass at $x = 0$ [conditions: $Q_n(x) \geq 0$ for $0 \leq x \leq 2\pi$, $\int_0^{2\pi} Q_n(t) dt = 1$, and for each open neighborhood I of 0,

Q_n converges to 0 uniformly on $\mathbb{T} \setminus I$. For $f \in C(\mathbb{T})$, define P_n as the convolution $f * Q_n$. In other words,

$$P_n(x) := \int_0^{2\pi} f(t)Q_n(x-t) dt = \int_0^{2\pi} f(x-t)Q_n(t) dt.$$

A standard argument shows that $\|P_n - f\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Now if $f \in V$, then P_n has only those harmonics e^{ikx} for which $k \in I$, since other harmonics will be killed by integration. Hence $P_n \in S$ for all n .

Second solution. We use the classical Theorem of Fejér ([K], Theorem 3.1): The arithmetic means of the partial sums of the Fourier series of a continuous function converge uniformly to the function. This means that if $f \in C(\mathbb{T})$, $S_n := \sum_{k=-n}^n \hat{f}(k)e^{ikx}$, and $P_n = (S_0 + S_1 + \dots + S_{n-1})/n$, then $\|P_n - f\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Now if $f \in V$, each S_n has only those harmonics e^{ikx} for which $k \in I$. So the same must be true for each P_n . Hence, again, $P_n \in S$ for all n .

D11. As in problem D10, we use the Theorem of Fejér ([K], Theorem 3.1): If $f \in C(\mathbb{T})$, $S_n := \sum_{k=-n}^n \hat{f}(k)e^{ikx}$, and $P_n = (S_0 + S_1 + \dots + S_{n-1})/n$, then $\|P_n - f\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. All we have to show is that for a fixed n , the Fourier coefficients of P_n are no larger in absolute value than the corresponding coefficients of f . But this is immediate since a brief computation shows

$$P_n(x) = \sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} \hat{f}(k)e^{ikx},$$

so that

$$|\hat{P}_n(k)| = \begin{cases} \frac{n-|k|}{n} |\hat{f}(k)| & |k| < n \\ 0 & |k| \geq n \end{cases}$$

D12. First note that

$$|\hat{f}(0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(x) dx \right| \leq \|f\|_1. \quad (20)$$

On the other hand, Integration by Parts shows that the n -th Fourier coefficient of f' is $in\hat{f}(n)$, so by Parseval's Identity

$$\sum_{-\infty}^{\infty} |n\hat{f}(n)|^2 = \|f'\|_2^2.$$

It follows from Cauchy-Schwarz that

$$\sum_{n \neq 0} |\hat{f}(n)| = \sum_{n \neq 0} \frac{1}{|n|} |n \hat{f}(n)| \leq \left(2 \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \|f'\|_2 = \frac{\pi}{\sqrt{3}} \|f'\|_2 \quad (21)$$

since $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. Adding (20) and (21), we obtain the desired inequality.

E. Basic Complex Analysis

E1. Let $P(z) = 0$ and $R := |z|$. Then

$$\begin{aligned} R^n = |z|^n &\leq \sum_{j=0}^{n-1} |a_j| |z|^j \\ &\leq \left(\sum_{j=0}^{n-1} |a_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{n-1} R^{2j} \right)^{\frac{1}{2}} \\ &= (r^2 - 1)^{\frac{1}{2}} \left(\frac{R^{2n} - 1}{R^2 - 1} \right)^{\frac{1}{2}} \\ &< (r^2 - 1)^{\frac{1}{2}} \frac{R^n}{(R^2 - 1)^{\frac{1}{2}}} \end{aligned}$$

from which it follows that $R < r$.

E2. Let $z_k := r_k e^{i\theta_k}$, where $r_k \geq 0$ and $0 \leq \theta_k < 2\pi$. Let I_θ be the set of integers $1 \leq k \leq n$ such that $\cos(\theta - \theta_k) \geq 0$. Then

$$\left| \sum_{k \in I_\theta} z_k \right| = \left| \sum_{k \in I_\theta} z_k e^{-i\theta} \right| \geq \sum_{k \in I_\theta} \operatorname{Re}(r_k e^{i(\theta_k - \theta)}) = \sum_{k=1}^n r_k \cos^+(\theta - \theta_k),$$

where $\cos^+ x := \max\{\cos x, 0\}$. Now choose θ_0 so as to maximize the last sum and set $I := I_{\theta_0}$. It follows that $|\sum_{k \in I} z_k|$ is at least as large as the average value of the function $\theta \mapsto \sum_{k=1}^n r_k \cos^+(\theta - \theta_k)$ over $0 \leq \theta \leq 2\pi$. This average value is easily seen to be $(1/\pi) \sum_{k=1}^n r_k$ since for every k ,

$$\int_0^{2\pi} \cos^+(\theta - \theta_k) d\theta = 2.$$

E3. Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z - 1,$$

where the a_j are real. Since all roots are outside the unit disk, their product is at least one in absolute value. But this product is $(-1)^{n+1}$ by the above expression. Hence

all roots must belong to the unit circle. Now $P(0) = -1$ and $\lim_{x \rightarrow +\infty} P(x) = +\infty$ implies that there exists a real number $x > 0$ with $P(x) = 0$. The only possible value for x is 1, hence $P(1) = 0$.

E4. Let $f(t) := 1 - \sum_{k=1}^n z_k e^{2\pi i k t}$. Note that

$$\left| \int_0^1 f(t) dt \right| = \left| 1 - \sum_{k=1}^n z_k \int_0^1 e^{2\pi i k t} dt \right| = 1.$$

Now if $|f(t)| < 1$ for all $0 < t < 1$, then

$$1 = \left| \int_0^1 f(t) dt \right| \leq \int_0^1 |f(t)| dt < \int_0^1 dt = 1,$$

which is a contradiction.

E5. g is a rational map with $g^{-1}(\infty) = \{\infty\}$. There are several ways to see that g is a polynomial. For example, one can write g as a ratio P/Q of relatively prime polynomials and check that Q must be a constant. Or note that $g|_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$ is entire and $\lim_{z \rightarrow \infty} g(z) = \infty$, hence it has a pole at infinity by Casorati-Weierstrass.

E6. First solution. Define the meromorphic function $Q(z) := 1/P(z)$ and note that

$$\text{Res}[Q; a_j] = \frac{1}{\prod_{i \neq j} (a_j - a_i)} = \frac{1}{P'(a_j)}.$$

Since

$$\sum_{j=1}^n \text{Res}[Q; a_j] = \frac{1}{2\pi i} \int_{|z|=R} Q(z) dz$$

for large $R > 0$ by the Residue Theorem, it suffices to show that $\int_{|z|=R} Q(z) dz = 0$ for all large R . Note that this integral is independent of R if the circle $|z| = R$ contains all the a_j (Cauchy-Goursat Theorem). Take R so large that $|Q(z)| < 2/R^n$ if $|z| = R$. Then

$$\left| \int_{|z|=R} Q(z) dz \right| \leq R \int_0^{2\pi} |Q(Re^{i\theta})| d\theta \leq R \cdot 2\pi \cdot \frac{2}{R^n}.$$

As $R \rightarrow \infty$, the last expression tends to zero since $n \geq 2$, and we are done.

Second solution. Let $D_j(z) := \prod_{i \neq j} (z - a_i)$. Since the a_j are distinct, the n polynomials $\{D_j\}_{1 \leq j \leq n}$ are linearly independent in the space of all complex polynomials

of degree $\leq n - 1$. This space has dimension n , so $\{D_j\}_{1 \leq j \leq n}$ forms a basis for it. Therefore, there exist constants $\lambda_1, \dots, \lambda_n$ such that $\sum_{j=1}^n \lambda_j D_j \equiv 1$. In particular,

$$1 = \sum_{j=1}^n \lambda_j D_j(a_k) = \lambda_k D_k(a_k) = \lambda_k P'(a_k)$$

so that

$$\lambda_k = \frac{1}{P'(a_k)}.$$

But $\sum_{k=1}^n 1/P'(a_k) = \sum_{k=1}^n \lambda_k$ is the coefficient of z^{n-1} in $\sum_{j=1}^n \lambda_j D_j \equiv 1$, which is clearly zero.

E7. Let

$$P(z) = A(z - a_1) \cdots (z - a_n),$$

where $A \neq 0$ and $|a_j| < 1$ for all j . It is easy to see that

$$P^*(z) = \bar{A}(1 - \bar{a}_1 z) \cdots (1 - \bar{a}_n z).$$

If $P(w) + P^*(w) = 0$ for some $w \in \mathbb{C}$, then in particular $|P(w)| = |P^*(w)|$, so that

$$\prod_{j=1}^n |w - a_j| = \prod_{j=1}^n |1 - \bar{a}_j w|.$$

Since neither side of the above equation can be zero, it follows that $|B(w)| = 1$, where

$$B(z) := \prod_{j=1}^n \left(\frac{z - a_j}{1 - \bar{a}_j z} \right)$$

is a Blaschke product. Since $|a_j| < 1$ for all j , it is easy to see that $B^{-1}(\mathbb{D}) = \mathbb{D}$ and $B^{-1}(\mathbb{C} \setminus \mathbb{D}) = \mathbb{C} \setminus \mathbb{D}$. Hence $|w| = 1$.

F. Properties of Holomorphic and Harmonic Functions

F1. Fix a $z \in \mathbb{D}$, let $\delta := 1 - |z|$. By Cauchy's Formula, for any $0 < r < \delta$ we have

$$f(z) = \frac{1}{2\pi i} \int_{|w-z|=r} \frac{f(w)}{w-z} dw$$

so that

$$|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z + re^{i\theta})| d\theta,$$

or

$$r|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \sqrt{r} |f(z + re^{i\theta})| \cdot \sqrt{r} d\theta.$$

Integrating with respect to r from 0 to δ , we obtain

$$\begin{aligned}
\frac{1}{2}\delta^2|f(z)| &\leq \frac{1}{2\pi} \int_0^\delta \int_0^{2\pi} \sqrt{r}|f(z+re^{i\theta})| \cdot \sqrt{r} d\theta dr \\
&\leq \frac{1}{2\pi} \left(\int_0^\delta \int_0^{2\pi} r|f(z+re^{i\theta})|^2 d\theta dr \right)^{\frac{1}{2}} \left(\int_0^\delta \int_0^{2\pi} r d\theta dr \right)^{\frac{1}{2}} \\
&= \frac{1}{2\pi} \left(\int_{\mathbb{D}(z,\delta)} |f|^2 dx dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{D}(z,\delta)} dx dy \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2\pi} \|f\|_2 \sqrt{\pi} \delta
\end{aligned}$$

from which it follows that

$$|f(z)| \leq \frac{1}{\sqrt{\pi}\delta} \|f\|_2.$$

F2. Let $f(z) = \sum_{n=0}^\infty c_n z^n$ for $|z| < 1$ so that $f'(z) = \sum_{n=1}^\infty n c_n z^{n-1}$. Cauchy's Formula applied to f' shows that for every $r < 1$ and $n \geq 1$

$$c_n = \frac{1}{2\pi i n} \int_{|z|=r} \frac{f'(z)}{z^n} dz,$$

from which it follows that

$$\begin{aligned}
|c_n| &= \left| \frac{1}{2\pi n r^{n-1}} \int_0^{2\pi} f'(r e^{i\theta}) e^{-i(n-1)\theta} d\theta \right| \\
&\leq \frac{1}{2\pi n r^{n-1}} \int_0^{2\pi} |f'(r e^{i\theta})| d\theta \\
&\leq \frac{M}{2\pi n r^{n-1}}.
\end{aligned}$$

Letting $r \rightarrow 1$, we obtain $|c_n| \leq M/(2\pi n)$ for all $n \geq 1$. Now Lebesgue's Monotone Convergence Theorem shows that

$$\begin{aligned}
\int_0^1 |f(x)| dx &= \int_0^1 \left| \sum_{n=0}^\infty c_n x^n \right| dx \\
&\leq \sum_{n=0}^\infty |c_n| \int_0^1 x^n dx \\
&= |c_0| + \sum_{n=1}^\infty \frac{|c_n|}{n+1} \\
&\leq |c_0| + \frac{M}{2\pi} \sum_{n=1}^\infty \frac{1}{n(n+1)},
\end{aligned}$$

which proves the result since the last series converges to 1.

F3. Since holomorphic maps are orientation-preserving, the restriction of f to the real line must be increasing so that $f'(x) \geq 0$ for every $x \in \mathbb{R}$. If $f'(x) = 0$ for some $x \in \mathbb{R}$, then locally near x the function f acts like a power $z \mapsto z^n$ for some $n \geq 2$. Therefore, there is a topological sector $S \subset \mathbb{H}$ of angle $2\pi/n \leq \pi$, with vertex x , which maps to an entire punctured neighborhood of $f(x)$. This contradicts $f(S) \subset f(\mathbb{H}) \subset \mathbb{H}$.

F4. Consider the power series $e^z = \sum_{n=0}^{\infty} z^n/n!$ which converges locally uniformly on the entire plane. It follows that

$$I := \frac{1}{2\pi} \int_0^{2\pi} e^{2\zeta \cos \theta} d\theta = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(2\zeta)^n}{n!} \int_0^{2\pi} \cos^n \theta d\theta.$$

Repeated Integration by Parts yields

$$\int_0^{2\pi} \cos^n \theta d\theta = \begin{cases} 0 & n \text{ odd} \\ 2\pi \frac{(n-1) \cdot (n-3) \cdots 3 \cdot 1}{n \cdot (n-2) \cdots 4 \cdot 2} & n \text{ even} \end{cases}$$

Hence, setting $n = 2k$, we obtain

$$\begin{aligned} I &= \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{(2\zeta)^{2k}}{(2k)!} \cdot 2\pi \frac{(2k-1) \cdot (2k-3) \cdots 3 \cdot 1}{2k \cdot (2k-2) \cdots 4 \cdot 2} \\ &= \sum_{k=0}^{\infty} \frac{(2\zeta)^{2k}}{(2k \cdot (2k-2) \cdots 4 \cdot 2)^2} \\ &= \sum_{k=0}^{\infty} \left(\frac{\zeta^k}{k!} \right)^2. \end{aligned}$$

F5. Let f be an entire function whose real part is u and consider $g := e^f$. It follows from the assumption on u that

$$|g(z)| = e^{u(z)} \leq e^{a|\log|z||+b} = e^b |z|^a$$

if $|z| > 1$. It follows from Cauchy's estimates that for all $n \geq 0$ and all $r > 1$,

$$|g^{(n)}(0)| \leq \frac{n! e^b r^a}{r^n}.$$

If $n > a$, by letting $r \rightarrow +\infty$, we see that $g^{(n)}(0) = 0$. It follows that g is a polynomial (of degree at most $[a]$). But $g = e^f$ never vanishes, so it must be a non-zero constant. It follows that f and u are constants as well.

F6. Using the complex differential operators ∂ and $\bar{\partial}$, it is easy to see that the Jacobian determinant of h at any point z is equal to $|\partial h(z)|^2 - |\bar{\partial} h(z)|^2$. If h fails to be one-to-one on any neighborhood of 0, it follows from the Inverse Function Theorem that $|\partial h(0)| = |\bar{\partial} h(0)|$. But

$$\partial h(z) = \partial f(z) = f'(z) \quad \bar{\partial} h(z) = \bar{\partial} \bar{g}(z) = \overline{g'(z)}$$

since $\bar{\partial} \bar{g}(z) = \bar{\partial} f(z) = 0$. Therefore, we must have $|f'(0)| = |g'(0)|$.

F7. First we show that for every compact set $K \subset U$ there exists an $M = M(K) > 0$ such that $|f(t, z) - f(t, w)|/|z - w| \leq M$ whenever z, w are distinct points of K and $t \in X$. Choose $\delta = \delta(K) > 0$ such that $\mathbb{D}(z, \delta) \subset U$ if $z \in K$. Fix distinct points z and w in K . If $|z - w| \geq \delta$, then

$$\left| \frac{f(t, z) - f(t, w)}{z - w} \right| \leq \frac{2\|f\|_\infty}{\delta}.$$

If, on the other hand, $|z - w| < \delta$, let the Jordan curve γ be the boundary of the union of the disks $\mathbb{D}(z, \delta) \cup \mathbb{D}(w, \delta)$, oriented counterclockwise. Since by Cauchy's Formula

$$f(t, z) = \frac{1}{2\pi i} \int_\gamma \frac{f(t, \zeta)}{\zeta - z} d\zeta \quad \text{and} \quad f(t, w) = \frac{1}{2\pi i} \int_\gamma \frac{f(t, \zeta)}{\zeta - w} d\zeta,$$

we have

$$\frac{f(t, z) - f(t, w)}{z - w} = \frac{1}{2\pi i} \int_\gamma \frac{f(t, \zeta)}{(\zeta - z)(\zeta - w)} d\zeta.$$

It follows that

$$\left| \frac{f(t, z) - f(t, w)}{z - w} \right| \leq \frac{1}{2\pi} \cdot \|f\|_\infty \cdot \text{length}(\gamma) \cdot \frac{1}{\delta^2} \leq \frac{2\|f\|_\infty}{\delta}$$

since the length of γ is less than $4\pi\delta$. Therefore, in either case one can take $M := 2\|f\|_\infty/\delta$ and this proves the first claim.

Now fix a $z \in U$, let $K \subset U$ be a compact neighborhood of z and find the corresponding $M = M(K)$ as above. Write

$$\frac{\varphi(z) - \varphi(w)}{z - w} = \int_X \frac{f(t, z) - f(t, w)}{z - w} d\mu(t).$$

For each $t \in X$, the integrand is bounded by M if w is restricted to K , and tends to $(\partial f/\partial z)(t, z)$ as $w \rightarrow z$. So by Lebesgue's Dominated Convergence Theorem, $\varphi'(z)$ exists and

$$\varphi'(z) = \int_X \frac{\partial f}{\partial z}(t, z) d\mu(t).$$

F8. *First solution.* g is holomorphic since

$$g(z) = \int_0^1 \sum_{n=0}^{\infty} \frac{(tz)^n}{n!} f(t) dt = \sum_{n=0}^{\infty} \frac{\int_0^1 t^n f(t) dt}{n!} z^n.$$

(Interchanging the sum and the integral is legitimate because the power series of the exponential function converges locally uniformly in the plane.) If g is identically zero, then $\int_0^1 t^n f(t) dt = 0$ for all $n \geq 0$. Hence $\int_0^1 P(t)f(t) dt = 0$ for all polynomials P . By Weierstrass Approximation Theorem, $\int_0^1 h(t)f(t) dt = 0$ for all continuous functions $h : [0, 1] \rightarrow \mathbb{C}$. By Lusin's Theorem ([R2], Theorem 2.24), $\int_0^1 \chi_E(t)f(t) dt = 0$ for all measurable sets $E \subset [0, 1]$. We conclude that $f = 0$ almost everywhere.

Second solution. If $g \equiv 0$, then in particular $g(2\pi in) = 0$ for all $n \in \mathbb{Z}$. This means that the Fourier coefficients of the \mathbb{Z} -periodic extension of f are all zero. Since trigonometric polynomials are dense in the space of continuous functions on $[0, 1]$, we obtain $\int_0^1 h(t)f(t) dt = 0$ for all continuous functions $h : [0, 1] \rightarrow \mathbb{C}$. The rest of the proof is now similar to the first solution. (Note that if we knew $f \in L^2[0, 1]$, we could immediately conclude $f = 0$ almost everywhere from Parseval.)

F9. *First solution.* By Harnack Inequality ([R2], Theorem 11.11), we have

$$\left(\frac{1-r}{1+r}\right) u(0) \leq u(r) \leq \left(\frac{1+r}{1-r}\right) u(0)$$

which implies

$$\frac{1}{3} \leq u\left(\frac{1}{2}\right) \leq 3.$$

This method gives us a little chance of discovering the optimal examples which show that these bounds are sharp. See the next solution for these examples.

Second solution. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be a holomorphic map with real part u and $f(0) = 1$. Since $z \mapsto w = (1-z)/(1+z)$ maps the right half-plane conformally onto the unit disk, it follows that

$$g(z) := \frac{1-f(z)}{1+f(z)}$$

is a holomorphic map $\mathbb{D} \rightarrow \mathbb{D}$ with $g(0) = 0$. By Schwarz Lemma, $|g(1/2)| \leq 1/2$, which means $f(1/2)$ belongs to the disk $|z - 5/3| \leq 4/3$. This clearly implies

$$\frac{1}{3} \leq u\left(\frac{1}{2}\right) \leq 3.$$

The extremal examples correspond to rotations $g(z) = z$ and $g(z) = -z$, leading respectively to

$$u(z) = \operatorname{Re} \left(\frac{1-z}{1+z} \right), u \left(\frac{1}{2} \right) = \frac{1}{3}$$

and

$$u(z) = \operatorname{Re} \left(\frac{1+z}{1-z} \right), u \left(\frac{1}{2} \right) = 3.$$

F10. Let $|f| = C$ on the boundary of D and fix $a \in D$. Since f has no zeros in D , by the Maximum Principle applied to $1/f$, $|f(a)| \geq \min_{z \in \partial D} |f(z)| = C$. On the other hand, by the Maximum Principle applied to f , $|f(a)| \leq \max_{z \in \partial D} |f(z)| = C$. Hence $|f(a)| = C$. It follows again from the Maximum Principle (or the fact that non-constant holomorphic maps are open) that f is constant throughout D , hence throughout U .

F11. We construct a sequence of polynomials Q_n such that $Q_n(z) \rightarrow 1$ if $z \geq 0$ and $Q_n(z) \rightarrow 0$ otherwise. Then the sequence R_n defined by $R_n(z) := Q_n(z)Q_n(-z)$ converges to 1 if $z = 0$ and to 0 otherwise. Finally, we can define $P_n(z) := R_n(z)/R_n(0)$ which has the desired property.

Let U_n be the union of the following four open rectangles:

$$(-n-1, n+1) \times (-n-1, -\frac{3}{n})$$

$$(-n-1, n+1) \times (\frac{3}{n}, n+1)$$

$$(-n-1, -\frac{3}{n}) \times (-n-1, n+1)$$

$$V_n := (-\frac{2}{n}, n+1) \times (-\frac{2}{n}, \frac{2}{n}).$$

Evidently U_n is open, has two connected components, and $\mathbb{C} \setminus U_n$ is connected. Let $K_n \subset U_n$ be the compact set

$$K_n := \{z \in U_n : \operatorname{dist}(z, \partial U_n) \geq \frac{1}{n}\}.$$

Finally, define $f_n : U_n \rightarrow \mathbb{C}$ to be 1 on V_n and 0 otherwise. Clearly f_n is holomorphic on U_n . By the Theorem of Runge ([R2], Theorem 13.6), we can find a polynomial Q_n such that

$$\sup_{z \in K_n} |Q_n(z) - f_n(z)| \leq \frac{1}{n}.$$

It follows that $Q_n(z) \rightarrow 1$ for $z \geq 0$, and $Q_n(z) \rightarrow 0$ otherwise.

F12. Since $u(x + iy) \rightarrow 0$ as $y \rightarrow 0$, we can extend u to the complex plane by

$$U(z) := \begin{cases} u(z) & \operatorname{Im} z \geq 0 \\ -u(\bar{z}) & \operatorname{Im} z \leq 0 \end{cases}$$

which is harmonic in \mathbb{C} and satisfies $|U(z)| \leq M|\operatorname{Im} z|$. Let $F : \mathbb{C} \rightarrow \mathbb{C}$ be an entire function whose real part is U . By Poisson's Formula,

$$F(z) = \frac{1}{2\pi i} \int_{|w|=R} \left(\frac{w+z}{w-z} \right) U(w) \frac{dw}{w} + \text{constant}$$

if $|z| < R$. It follows that

$$F'(z) = \frac{1}{\pi i} \int_{|w|=R} \frac{U(w)}{(w-z)^2} dw.$$

Fix a $z \in \mathbb{C}$ and choose $R > 0$ so large that $R > 2|z|$. Then

$$|F'(z)| \leq \frac{1}{\pi} (2\pi R) \frac{MR}{R^2/4} = 8M.$$

It follows from Liouville's Theorem that F' is a constant function, hence F will be an affine map. We conclude that u must be of the form $x + iy \mapsto ay$ for some $0 \leq a \leq M$.

F13. (a) $M(r) \leq M_1(r)$ simply follows from Triangle Inequality. On the other hand, Cauchy's estimates show that $|c_n| \leq M(R)/R^n$. Hence

$$M_1(r) \leq \sum_{n=0}^{\infty} M(R) \left(\frac{r}{R} \right)^n = \frac{R}{R-r} M(R).$$

(b) First by Cauchy-Schwarz,

$$M_1(r) = \sum_{n=0}^{\infty} |c_n| R^n \left(\frac{r}{R} \right)^n \leq \left(\sum_{n=0}^{\infty} |c_n|^2 R^{2n} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \left(\frac{r}{R} \right)^{2n} \right)^{\frac{1}{2}},$$

or

$$M_1(r) \leq M_2(R) \left(\frac{R^2}{R^2 - r^2} \right)^{\frac{1}{2}}.$$

This proves the inequality $\sqrt{R^2 - r^2} M_1(r)/R \leq M_2(R)$. On the other hand, Parseval's Identity for Fourier series shows that

$$M_2(R) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}}$$

which proves the inequality $M_2(R) \leq M(R)$.

F14. Let $f = u + iv$ and $n \geq 0$. By Cauchy's Formula

$$c_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz = \frac{1}{2\pi r^n} \int_0^{2\pi} (u + iv)(re^{i\theta})e^{-in\theta} d\theta.$$

Similarly, since $\int_{|z|=r} f(z)z^n dz = 0$, we obtain

$$0 = \frac{1}{2\pi r^n} \int_0^{2\pi} (u + iv)(re^{i\theta})e^{in\theta} d\theta.$$

Adding the first formula to the conjugate of the second formula, we get

$$c_n r^n = \frac{1}{\pi} \int_0^{2\pi} u(re^{i\theta})e^{-in\theta} d\theta$$

from which it follows that

$$|c_n| r^n \leq \frac{1}{\pi} \int_0^{2\pi} |u(re^{i\theta})| d\theta. \quad (22)$$

On the other hand, as a harmonic function on \mathbb{D} , u has the mean value property:

$$2u(0) = \frac{1}{\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta. \quad (23)$$

Adding (22) and (23), we obtain

$$|c_n| r^n + 2u(0) \leq \frac{1}{\pi} \int_0^{2\pi} (|u| + u)(re^{i\theta}) d\theta = \frac{2}{\pi} \int_0^{2\pi} u^+(re^{i\theta}) d\theta,$$

where $u^+ := \max\{u, 0\}$. Noting that $u^+(re^{i\theta}) \leq \max\{A(r), 0\}$ for every θ , it follows that the last integral is bounded above by $4 \max\{A(r), 0\}$.

F15. It suffices to prove that for every $\varepsilon > 0$, the set $G := \{z \in U : |f(z)| > M + \varepsilon\}$ is empty. Assuming the contrary, the non-vacuous open set G must be compactly contained in U . In fact, if $a \in \partial U$ belongs to \overline{G} , there must be a sequence $z_n \in G$ converging to a along which $|f(z_n)| > M + \varepsilon$. Hence $\limsup_{n \rightarrow \infty} |f(z_n)| \geq M + \varepsilon$ which contradicts the assumption on f . Thus $\overline{G} \subset U$. Since $z \mapsto |f(z)|$ is continuous, on the boundary of G we have $|f| = M + \varepsilon$. It follows from the Maximum Principle that $|f| \leq M + \varepsilon$ throughout \overline{G} . This contradicts the definition of G .

F16. Define $\varphi(z) := \sup_n |f_n(z)|$. Note that φ is lower semicontinuous and $\varphi < +\infty$ throughout U . First we show that for every closed disk $D \subset U$ there exists a closed subdisk $K \subset D$ on which φ is uniformly bounded by some constant $M(D, K) > 0$. Fix such D and define

$$K_n := \{z \in D : |\varphi(z)| \leq n\}.$$

Each K_n is closed because φ is lower semicontinuous. Since $D = \bigcup_n K_n$, by Baire's Category Theorem some K_n must have non-empty interior. Now we can choose our subdisk K anywhere in the interior of that K_n and set $M(D, K) := n$.

We show that f is holomorphic in the interior of K . Since D was chosen arbitrarily, this proves that the set of points where f is holomorphic is dense. Let γ denote the boundary circle of K oriented counterclockwise. By Cauchy's Formula,

$$f_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(w)}{w - z} dw$$

for all z in the interior of K and all n . The integrand is bounded in absolute value by $M(D, K)/\text{dist}(z, \partial K)$. Hence Lebesgue's Dominated Convergence Theorem implies

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw$$

for all z in the interior of K . It is a classical fact that a function is holomorphic on the set of points where it is representable by such an integral formula (see e.g., [R2], Theorem 10.7).

F17. Consider the holomorphic function $f = u + iv$ and let $z = re^{i\theta}$ where $r < R$. By Poisson's Formula, for any s with $r < s < R$, we have

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{se^{it} + z}{se^{it} - z} u(se^{it}) dt.$$

Taking the imaginary part of this expression gives

$$v(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Im} \left(\frac{se^{it} + z}{se^{it} - z} \right) u(se^{it}) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2sr \sin(\theta - t)}{s^2 + r^2 - 2sr \cos(\theta - t)} u(se^{it}) dt.$$

To estimate this last integral, without loss of generality, we may assume $\theta = 0$. It follows that

$$|v(z)| \leq \frac{M}{2\pi} \int_{-\pi}^{\pi} \left| \frac{2sr \sin t}{s^2 + r^2 - 2sr \cos t} \right| dt = \frac{M}{\pi} \int_0^{\pi} \frac{2sr \sin t}{s^2 + r^2 - 2sr \cos t} dt.$$

A simple change of variable $\phi = s^2 + r^2 - 2sr \cos t$ shows that

$$|v(z)| \leq \frac{M}{\pi} \int_{(s-r)^2}^{(s+r)^2} \frac{d\phi}{\phi} = \frac{2M}{\pi} \log \left(\frac{s+r}{s-r} \right).$$

Letting $s \rightarrow R$, we obtain the required inequality.

To show that $|v|$ can actually be unbounded even when u is bounded, note that by the Riemann Mapping Theorem there exists a conformal map $f = u + iv : \mathbb{D} \rightarrow \{z : -1 < \operatorname{Re}(z) < 1\}$ with $f(0) = 0$. Also see problem G11 for a similar result.

F18. (a) Fix $-\infty < t_1 < t_2 < \log R$ and $0 < \lambda < 1$. Let $h(z) := \alpha \log |z| + \beta$, where the constants α and β are chosen in such a way that

$$h(e^{t_1}) = M(t_1), \quad h(e^{t_2}) = M(t_2).$$

Let A denote the annulus $e^{t_1} < |z| < e^{t_2}$. Since $u \leq h$ on the boundary of A and h is harmonic, the Maximum Principle for subharmonic functions implies that $u \leq h$ throughout A . Therefore,

$$\begin{aligned} M(\lambda t_1 + (1 - \lambda)t_2) &\leq h(e^{\lambda t_1 + (1 - \lambda)t_2}) \\ &= \lambda(\alpha t_1 + \beta) + (1 - \lambda)(\alpha t_2 + \beta) \\ &= \lambda M(t_1) + (1 - \lambda)M(t_2), \end{aligned}$$

proving that M is convex.

(b) Let $u : \mathbb{C} \rightarrow \mathbb{R}$ be bounded. Then M is a bounded function on the real line which is convex by (a). So M must be constant. It easily follows that u is constant.

F19. We show that φ is subharmonic. This, in particular, implies that φ satisfies the Maximum Principle on U . Clearly φ is continuous. Fix $z \in U$ and any $r > 0$ such that $\overline{\mathbb{D}(z, r)} \subset U$. Then

$$\begin{aligned} \varphi(z) = \sum_{j=1}^n |u_j(z)| &= \sum_{j=1}^n \frac{1}{2\pi} \left| \int_0^{2\pi} u_j(z + re^{it}) dt \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{j=1}^n |u_j(z + re^{it})| \right) dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + re^{it}) dt. \end{aligned}$$

This is the characterizing property of subharmonic functions.

F20. If f has infinitely many zeros in \mathbb{D} , either it is identically zero or it has a sequence of zeros approaching $\partial\mathbb{D}$. In either case, the result is trivial. So let us assume that f has only finitely many zeros a_1, a_2, \dots, a_k in \mathbb{D} . Consider the Blaschke

product

$$B(z) = \prod_{j=1}^k \left(\frac{z - a_j}{1 - \overline{a_j}z} \right)$$

with zeros at the a_j . (As usual, if some a_j is a multiple root of f , we repeat it in B the same number of times.) Then the function B/f is holomorphic in \mathbb{D} . Assume by way of contradiction that there is no sequence $z_n \in \mathbb{D}$ with $|z_n| \rightarrow 1$ such that $\{f(z_n)\}$ is bounded. Then $|f(z)| \rightarrow +\infty$ as $|z| \rightarrow 1$. Since $|B(z)| \rightarrow 1$ as $|z| \rightarrow 1$, it follows that the holomorphic function $B/f : \mathbb{D} \rightarrow \mathbb{C}$ tends to zero as $|z| \rightarrow 1$. By the generalized Maximum Principle (problem F15), we must have $B/f \equiv 0$, which is a contradiction.

F21. Let $\zeta := g(z)$. Then the relation $h(\zeta) = f(z)$ defines h uniquely at $\zeta \in g(U)$ since if $\zeta = g(w)$ also, then $f(z) = f(w)$. If g is constant, then f is also a constant and we can take h as a translation. So assume g is not constant. Let $C \subset U$ be the discrete set of critical points of g , i.e., $z \in C$ if and only if $g'(z) = 0$. We choose a $\zeta_0 \in g(U)$ and show that h is holomorphic at ζ_0 .

Case 1. $\zeta_0 \in g(U \setminus C)$. Then there exists a point $z_0 \in U \setminus C$ with $g(z_0) = \zeta_0$ and open neighborhoods D of z_0 and D' of ζ_0 such that $g : D \xrightarrow{\cong} D'$ is a conformal isomorphism. It follows that h is holomorphic near ζ_0 since $h(\zeta) = f(g^{-1}(\zeta))$ throughout D' .

Case 2. $g^{-1}(\zeta_0) \subset C$. Choose $z_0 \in C$ such that $g(z_0) = \zeta_0$ and open neighborhood D of z_0 such that $D \cap C = \{z_0\}$. Set $D' := g(D)$, which is an open neighborhood of ζ_0 . Note that by *Case 1*, h is holomorphic throughout $D' \setminus \{\zeta_0\}$. As $\zeta \in D' \setminus \{\zeta_0\}$ approaches ζ_0 , $h(\zeta)$ approaches the values of f at (any of) the g -preimages of ζ_0 . In particular, h stays bounded. Hence ζ_0 is a removable singularity of h .

F22. Fix z with $|z| = r$ and let $w := f(z) \in \Gamma_r$. The tangent direction to Γ_r at w is given by the vector $v := izf'(z)$. Hence the required distance d will be $|w| \sin \theta$, where θ is the angle between the vectors w and v ; in other words

$$\frac{w}{|w|} e^{i\theta} = \frac{v}{|v|} \quad \text{or} \quad \sin \theta = \operatorname{Im} \left(\frac{v|w|}{w|v|} \right).$$

It follows that

$$\begin{aligned}
 d &= |w| \operatorname{Im} \left(\frac{v|w|}{w|v|} \right) \\
 &= \frac{1}{|v|} \operatorname{Im}(v\bar{w}) \\
 &= \frac{\operatorname{Re}(zf'(z)\overline{f(z)})}{|zf'(z)|}.
 \end{aligned}$$

F23. Let us compute the curvature $\kappa(w)$ of Γ_r at a point $w = f(z)$. We parametrize Γ_r by $w(t) = f(z(t))$, where $z(t) = re^{it}$ for $0 \leq t \leq 2\pi$. Clearly the tangent vector v to Γ_r is given by $v(t) = dw/dt = izf'(z)$. By definition,

$$\kappa(w) = \frac{d \arg v}{ds},$$

where ds is the arclength element along Γ_r . Note that $ds/dt = |v| = |zf'(z)|$. We compute

$$\begin{aligned}
 \kappa(w) &= \frac{\frac{d}{dt} \arg v}{\frac{ds}{dt}} \\
 &= \frac{\frac{d}{dt} \operatorname{Im}[\log(izf'(z))]}{|zf'(z)|} \\
 &= \frac{\operatorname{Im} \left[\frac{d}{dz} (\log(izf'(z))) \cdot \frac{dz}{dt} \right]}{|zf'(z)|} \\
 &= \frac{\operatorname{Im} \left[\frac{izf''(z) + if'(z)}{izf'(z)} \cdot iz \right]}{|zf'(z)|} \\
 &= \frac{\operatorname{Re} \left[\frac{zf''(z)}{f'(z)} + 1 \right]}{|zf'(z)|}.
 \end{aligned}$$

Now Γ_r is a strictly convex curve if and only if $\kappa(w) > 0$ for all $w \in \Gamma_r$, and this is equivalent to the required condition.

F24. Every $f \in \mathcal{F}$ has a power series expansion of the form

$$f(z) = a_3z^3 + a_4z^4 + \dots$$

so that $\tilde{f}(z) := f(z)/z^3$ is a holomorphic function on \mathbb{D} . If $r < 1$ and $|z| = r$, then $|\tilde{f}(z)| \leq 1/r^3$, so by the Maximum Principle $|\tilde{f}(z)| \leq 1/r^3$ if $|z| \leq r$. Letting $r \rightarrow 1$, it follows that $|\tilde{f}(z)| \leq 1$ for $z \in \mathbb{D}$, or in other words

$$|f(z)| \leq |z|^3 \quad (f \in \mathcal{F}, z \in \mathbb{D}). \quad (24)$$

Note that if equality occurs in (24) for some $z \in \mathbb{D}$, then $f(z) = \lambda z^3$ with $|\lambda| = 1$.

It follows from (24) that $M := \sup_{f \in \mathcal{F}} |f(1/2)| \leq 1/8$. Since the cubic map $g(z) = z^3$ is already in \mathcal{F} , we have $M = 1/8$. Finally, let $f \in \mathcal{F}$ also satisfy $f(1/2) = 1/8$. Then equality occurs in (24) for $z = 1/2$, so $f(z) = \lambda g(z)$. Substituting $z = 1/2$, it follows that $\lambda = 1$, hence $f \equiv g$.

G. Schwarz Lemma and Conformal Maps

G1. Since P is univalent in \mathbb{D} , the derivative P' never vanishes inside \mathbb{D} , so every critical point c_j of P has absolute value at least 1. Hence the product $c_1 \cdots c_{n-1}$ is at least 1 in absolute value. But $P'(z) = 1 + 2a_2z + \cdots + na_nz^{n-1}$ so that

$$|c_1 \cdots c_{n-1}| = \frac{1}{|na_n|}.$$

It follows that $|a_n| \leq 1/n$.

To see that this is the best upper-bound, consider the polynomial $P(z) = z + \frac{1}{n}z^n$. If $P(z) = P(w)$ for distinct points $z, w \in \mathbb{D}$, then

$$z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1} = -n$$

which implies

$$n = |z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1}| \leq |z|^{n-1} + |z|^{n-2}|w| + \cdots + |z||w|^{n-2} + |w|^{n-1} < n,$$

which is a contradiction.

G2. First solution. For $0 < r < 1$ define $f_r(z) := f(rz)$ which holomorphically maps $\mathbb{D}(0, 1/r)$ to \mathbb{D} and has zeros at the z_k/r . Let r be close to 1 so that $|z_k| < r^2$ for all k . In particular, $z_k/r \in \mathbb{D}$. Consider the Blaschke product

$$B(z) := \prod_{k=1}^n \left(\frac{z - \frac{z_k}{r}}{1 - \frac{\overline{z_k}}{r}z} \right)$$

and let $g(z) := f_r(z)/B(z)$. Clearly g is holomorphic on $\mathbb{D}(0, 1/r)$ by the choice of r and B . On the boundary of the unit disk, $|B| = 1$ and $|f_r| < 1$, so by the Maximum Principle, $|g(z)| \leq 1$ throughout \mathbb{D} . This means

$$|f_r(z)| \leq \prod_{k=1}^n \left| \frac{z - \frac{z_k}{r}}{1 - \frac{\overline{z_k}}{r}z} \right|$$

for all $z \in \mathbb{D}$. Letting $r \rightarrow 1$, it follows that

$$|f(z)| \leq \prod_{k=1}^n \left| \frac{z - z_k}{1 - \overline{z_k}z} \right|.$$

Second solution. We can simplify the above solution by using generalized Maximum Principle (problem F15) as follows: Consider the Blaschke product

$$B(z) := \prod_{k=1}^n \left(\frac{z - z_k}{1 - \overline{z_k}z} \right)$$

and the quotient $g := f/B$ which is holomorphic in \mathbb{D} . Since $\lim_{|z| \rightarrow 1} |B(z)| = 1$ and $\limsup_{|z| \rightarrow 1} |f(z)| \leq 1$, we conclude that

$$\limsup_{|z| \rightarrow 1} |g(z)| \leq 1,$$

so that $|g(z)| \leq 1$ by problem F15.

G3. Let $f = u + iv$ and $0 < r < 1$. By Cauchy's Formula

$$f^{(n)}(0) = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta$$

or

$$c_n = \frac{1}{2\pi r^n} \int_0^{2\pi} (u + iv)(re^{i\theta}) e^{-in\theta} d\theta. \quad (25)$$

On the other hand, $z \mapsto f(z)z^n$ is holomorphic on \mathbb{D} if $n \geq 0$, so that

$$0 = \frac{1}{2\pi r^n} \int_0^{2\pi} (u + iv)(re^{i\theta}) e^{in\theta} d\theta. \quad (26)$$

Subtracting the conjugate of (26) from (25), we obtain

$$c_n = \frac{i}{\pi r^n} \int_0^{2\pi} v(re^{i\theta}) e^{-in\theta} d\theta.$$

But

$$c_n = \operatorname{Re}(c_n) = \frac{1}{\pi r^n} \int_0^{2\pi} v(re^{i\theta}) \sin n\theta d\theta,$$

which gives

$$|c_n| \leq \frac{1}{\pi r^n} \int_0^{2\pi} |v(re^{i\theta})| |\sin n\theta| d\theta \leq \frac{n}{\pi r^n} \int_0^{2\pi} |v(re^{i\theta})| |\sin \theta| d\theta$$

because $|\sin n\theta| \leq n|\sin \theta|$. Since c_n is real for all n and f is univalent, f preserves the real line so that $v(z) > 0$ if $\operatorname{Im}(z) > 0$ and $v(z) < 0$ if $\operatorname{Im}(z) < 0$. We conclude that

$$|c_n| \leq \frac{n}{\pi r^n} \int_0^{2\pi} v(re^{i\theta}) \sin \theta d\theta = \frac{n}{\pi r^n} (\pi r c_1) = \frac{nc_1}{r^{n-1}}.$$

Since $c_1 = 1$, by letting $r \rightarrow 1$, we obtain $|c_n| \leq n$.

G4. Let $|f(z)| \leq M$ for $z \in \mathbb{H}$. Consider the conformal isomorphism $\varphi : \mathbb{H} \rightarrow \mathbb{D}$ given by $\varphi(z) = (i - z)/(i + z)$. The mapping $g : \mathbb{D} \rightarrow \mathbb{D}$ defined by

$$g(z) := \frac{1}{M}f(\varphi^{-1}(z))$$

is holomorphic and fixes $z = 0$. By Schwarz Lemma, $|g(-1/3)| \leq 1/3$. It easily follows that

$$|f(2i)| \leq \frac{M}{3}.$$

This upper bound is sharp since it is achieved when g is a rotation.

On the other hand, the a priori lower bound on $|f(2i)|$ is evidently zero. To see this, just take any holomorphic map $h : \mathbb{D} \rightarrow \mathbb{D}$ with $h(0) = h(-1/3) = 0$ and define $f := h \circ \varphi$.

G5. This is a special case of problem G2 with $z_1 = 0, z_2 = r, z_3 = -r$:

$$|f(z)| \leq |z| \left| \frac{z - r}{1 - rz} \right| \left| \frac{z + r}{1 + rz} \right| = |z| \left| \frac{z^2 - r^2}{1 - r^2 z^2} \right|.$$

In particular, by substituting $z = r/2$, we obtain

$$\left| f\left(\frac{r}{2}\right) \right| \leq \frac{3}{2} \left(\frac{r^3}{4 - r^4} \right).$$

G6. Set $W := \{z \in \mathbb{C} : |z| < 1 \text{ and } \text{Im}(z) > 0\}$. The Möbius transformation $w := (1 + z)/(1 - z)$ maps W conformally onto the first quadrant Q . The squaring map $\zeta := w^2$ maps Q conformally onto the upper half-plane \mathbb{H} . Finally, the Möbius transformation $\xi := (i - \zeta)/(i + \zeta)$ maps \mathbb{H} conformally onto the unit disk \mathbb{D} . The long composition $f : z \mapsto w \mapsto \zeta \mapsto \xi$ then maps W conformally onto \mathbb{D} . Explicitly,

$$f(z) = \frac{i - \left(\frac{1+z}{1-z}\right)^2}{i + \left(\frac{1+z}{1-z}\right)^2} = \frac{i(1-z)^2 - (1+z)^2}{i(1-z)^2 + (1+z)^2}.$$

G7. We prove that if $0 < r < 1$ and f maps the circle $|z| = r$ injectively, then $f : \mathbb{D}(0, r) \rightarrow \mathbb{C}$ is univalent. Let Γ_r denote the Jordan curve $f(\{z : |z| = r\})$. If $|a| < r$ and $f(a)$ belongs to the exterior component of Γ_r , then $\text{ind}_{f(a)}\Gamma_r = 0$, which by the Argument Principle means f never takes on the value $f(a)$ for $|z| < r$, a contradiction. Therefore $f(a)$ belongs to the interior component of Γ_r , hence $\text{ind}_{f(a)}\Gamma_r = 1$. By another application of the Argument Principle, it follows that $f(a)$ is taken exactly once in $|z| < r$. Hence f restricted to $\mathbb{D}(0, r)$ is one-to-one.

G8. *First solution.* Fix $z \in \mathbb{D}$ and use the notation φ_p for the automorphism $w \mapsto (w - p)/(1 - \bar{p}w)$ of the unit disk. The map

$$g := \varphi_{f(z)} \circ f \circ \varphi_z^{-1} : \mathbb{D} \rightarrow \mathbb{D}$$

fixes the origin, so by Schwarz Lemma $|g'(0)| \leq 1$. A simple computation shows

$$\varphi_p'(p) = \frac{1}{1 - |p|^2}.$$

Since by the Chain Rule

$$g'(0) = \varphi'_{f(z)}(f(z)) \cdot f'(z) \cdot (\varphi'_z(z))^{-1},$$

we obtain

$$\frac{1}{1 - |f(z)|^2} \cdot |f'(z)| \cdot (1 - |z|^2) \leq 1$$

or

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Second solution. This is basically the same solution but it uses a different, more invariant language. By Schwarz Lemma f contracts the hyperbolic metric $\rho(w) = |dw|/(1 - |w|^2)$ on the unit disk $|w| < 1$, i.e., $f^*\rho \leq \rho$, where $f^*\rho$ is the pull-back metric. But, if $w = f(z)$, then

$$f^*\rho(z) = \frac{|df(z)|}{1 - |f(z)|^2} = \frac{|f'(z)||dz|}{1 - |f(z)|^2}$$

which shows

$$\frac{|f'(z)||dz|}{1 - |f(z)|^2} \leq \frac{|dz|}{1 - |z|^2},$$

which is the inequality we needed.

G9. Let $g(z) := M^{-1}f(rz)$ and apply problem G2 on g :

$$|g(z)| \leq \prod_{j=1}^n \left| \frac{z - \frac{z_j}{r}}{1 - \frac{\bar{z}_j}{r}z} \right|.$$

In particular,

$$|g(0)| \leq \prod_{j=1}^n \frac{|z_j|}{r}$$

or

$$\frac{1}{M}|f(0)| \leq \frac{1}{r^n}|z_1 z_2 \dots z_n|,$$

which is the inequality we wanted to prove.

G10. The function $\psi := f^{-1} \circ g : \mathbb{D} \rightarrow \mathbb{D}$ is well-defined and $\psi(0) = 0$. By Schwarz Lemma, $|\psi(z)| \leq |z|$ for all $z \in \mathbb{D}$. If $|z| < r < 1$, it follows that $|f^{-1}(g(z))| < r$, or $g(z) \in f(\mathbb{D}(0, r))$. This proves $g(\mathbb{D}(0, r)) \subset f(\mathbb{D}(0, r))$.

G11. Set $W := \{z \in \mathbb{C} : -1 < \operatorname{Re}(z) < 1\}$. Define the conformal map $\varphi : W \xrightarrow{\cong} \mathbb{D}$ by

$$\varphi(z) := \frac{i - e^{\frac{i\pi}{2}(z+1)}}{i + e^{\frac{i\pi}{2}(z+1)}},$$

with the inverse

$$\varphi^{-1}(z) = \frac{2}{\pi i} \log \left(\frac{1-z}{1+z} \right),$$

where the branch of the logarithm maps 0 to 0. Then the composition $g := \varphi \circ f : \mathbb{D} \rightarrow \mathbb{D}$ is well-defined and $g(0) = 0$. By Schwarz Lemma, $|g(re^{i\theta})| \leq r$, or

$$|f(re^{i\theta})| \leq \max_{|z|=r} |\varphi^{-1}(z)| = |\varphi^{-1}(-r)| = \frac{2}{\pi} \log \left(\frac{1+r}{1-r} \right).$$

G12. If $z \neq w$, then $Q_\lambda(z) = Q_\lambda(w)$ if and only if $z + w = -\lambda$. When $|\lambda| \geq 2$, $z + w = -\lambda$ has no solutions for $z, w \in \mathbb{D}$. Hence for $|\lambda| \geq 2$, Q_λ will be univalent in \mathbb{D} .

On the other hand, assume $|\lambda| < 2$. Let $\varepsilon > 0$ be small so that $z := -(1 - \varepsilon)\lambda/2$ and $w := -(1 + \varepsilon)\lambda/2$ are both in the unit disk. Since $z + w = -\lambda$, we have $Q_\lambda(z) = Q_\lambda(w)$. Hence Q_λ will not be univalent on the unit disk.

G13. Define $g : \mathbb{D} \rightarrow \mathbb{D}$ by $g(z) := f^{-1}(\omega f(z))$. Since g is conformal and $g(0) = 0$, Schwarz Lemma implies that g is a rigid rotation. Since $g'(0) = \omega$, we must have $g(z) = \omega z$, or $f(\omega z) = \omega f(z)$. Using the power series expansion of f , we obtain

$$\sum_{n=0}^{\infty} c_n \omega^n z^n = \sum_{n=0}^{\infty} c_n \omega z^n.$$

It follows that $c_n(\omega^n - \omega) = 0$ for all n . Hence either $c_n = 0$ or $\omega^{n-1} = 1$. The second equality is possible only if $n - 1$ is a multiple of k , or in other words $n \equiv 1 \pmod{k}$.

G14. By the Uniformization Theorem U is holomorphically covered by the unit disk. Let $p : \mathbb{D} \rightarrow U$ be the covering map with $p(0) = a$. Consider the lift of f to the universal covering which fixes the origin, i.e., the holomorphic map $g : \mathbb{D} \rightarrow \mathbb{D}$ such that $g(0) = 0$ and $p \circ g = f \circ p$. Note that $g'(0) = f'(a)$.

(a) This follows from Schwarz Lemma since $|f'(a)| = |g'(0)| \leq 1$.

(b) If $|f'(a)| = 1$, then $|g'(0)| = 1$ so by Schwarz Lemma g is a rigid rotation by some angle $0 \leq \theta < 1$ such that $f'(a) = g'(0) = e^{2\pi i \theta}$. Note that f is a local isometry with respect to the hyperbolic metric on U (since p and g are local isometries). Hence by Schwarz Lemma $f : U \rightarrow U$ must be a covering map of some degree $d \geq 1$. We prove that $d = 1$ so f must be a conformal automorphism of U . Assume by way of contradiction that $d \geq 2$ and distinguish two cases: (i) θ is irrational. Since $d \geq 2$, there exists a $b \neq a$ with $f(b) = a$. If $b' \in \mathbb{D}$ is any point with $p(b') = b$, then $b' \neq 0$ and the points in the forward orbit of b' under the rotation by angle θ must all map to a under the projection p . Since this orbit is dense on the circle $|z| = |b'|$, this circle must map under p to the single point a , which is impossible. (ii) θ is rational. Then some iterate g^{on} must be the identity map. It follows that the same iterate f^{on} is the identity map of U , hence its degree d^n must be 1. This contradicts $d \geq 2$.

(c) If $f'(a) = 1$, then $g'(0) = 1$ so $g(z) = z$ by Schwarz Lemma. It follows that $f(z) = z$ for all $z \in U$.

(d) No. If $U = \mathbb{C}$, take $f(z) = \lambda z + z^2$ for $\lambda \in \mathbb{C}$ and consider the fixed point $a = 0$. Since one can choose λ arbitrarily, none of the implications (a)-(c) holds. If $U = \mathbb{C}^*$, take for example $f(z) = e^{\lambda(z-1)}$, where $\lambda \in \mathbb{C}$. Note that $f(1) = 1$ and $f'(1) = \lambda$, so it is easy to choose λ in a such a way that none of (a)-(c) holds.

G15. *First solution.* Fix $0 < s < r$, $a, b \in \mathbb{D}(0, s)$, and $0 < \lambda < 1$. Without loss of generality assume $f(0) = 0$ and $|a| \leq |b| \neq 0$. Define $g : \mathbb{D}(0, r) \rightarrow \mathbb{C}$ by

$$g(z) := \lambda f\left(\frac{a}{b}z\right) + (1 - \lambda)f(z),$$

which satisfies $g(0) = f(0) = 0$. If $z \in \mathbb{D}(0, r)$, then $az/b \in \mathbb{D}(0, r)$. Since $f(\mathbb{D}(0, r))$ is convex, $g(z)$ (as a convex combination of $f(z)$ and $f(az/b)$) will be in $f(\mathbb{D}(0, r))$. It follows that $g(\mathbb{D}(0, r)) \subset f(\mathbb{D}(0, r))$. By problem G10, we have $g(\mathbb{D}(0, s)) \subset f(\mathbb{D}(0, s))$. In particular,

$$g(b) = \lambda f(a) + (1 - \lambda)f(b) \in f(\mathbb{D}(0, s)).$$

Therefore, $f(\mathbb{D}(0, s))$ must be convex.

Second solution. We use problem F23. Since $f(\mathbb{D}(0, r))$ is convex, the real part of the holomorphic function $z \mapsto zf''(z)/f'(z)$ is larger than -1 when $|z| = r$. It follows that the function

$$u(z) := \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} \right) + 1$$

takes positive values on the circle $|z| = r$. Since $f'(z) \neq 0$ for $z \in \mathbb{D}$, u is a harmonic function throughout \mathbb{D} , so by the Maximum Principle u will be positive on the entire disk $\mathbb{D}(0, r)$, and in particular on the circle $|z| = s < r$. Another application of problem F23 then shows that $f(\mathbb{D}(0, s))$ is convex.

G16. First note that by the Change of Variable Formula, the area V_r can be expressed as

$$V_r = \int \int_{f(\mathbb{D}(0, r))} dx dy = \int_0^r \int_0^{2\pi} |f'(se^{i\theta})|^2 s ds d\theta.$$

On the other hand, the function $z \mapsto (f'(z))^2$ is holomorphic on \mathbb{D} , so by Cauchy's Formula

$$1 = (f'(0))^2 = \frac{1}{2\pi} \int_0^{2\pi} (f'(se^{i\theta}))^2 d\theta$$

for every $s < 1$, which implies

$$2\pi \leq \int_0^{2\pi} |f'(se^{i\theta})|^2 d\theta.$$

Multiplying by s and integrating from 0 to r , we obtain

$$2\pi \left(\frac{1}{2}r^2\right) \leq \int_0^r \int_0^{2\pi} |f'(se^{i\theta})|^2 s ds d\theta,$$

or $V_r \geq \pi r^2$.

G17. (a) Define the univalent function $g : \mathbb{D} \rightarrow \mathbb{C}$ by $g(z) := (f(z) - f(0))/f'(0)$, which satisfies $g(0) = 0$ and $g'(0) = 1$. By Koebe 1/4-Theorem, $\text{dist}(\partial g(\mathbb{D}), 0) \geq 1/4$. On the other hand, the assumption that $0 \notin f(\mathbb{D})$ implies that the straight line from 0 to $f(0)$ intersects the boundary $\partial f(\mathbb{D})$. Hence $\text{dist}(\partial f(\mathbb{D}), 0) \leq |f(0)|$. It follows that $\text{dist}(\partial g(\mathbb{D}), 0) \leq |f(0)|/|f'(0)|$, which proves $|f'(0)| \leq 4|f(0)|$.

(b) Fix a point $w \in \mathbb{D}$ and apply (a) to the univalent function $h(z) := f((z + w)/(1 + \bar{w}z))$. Since $h'(0) = (1 - |w|^2)f'(w)$, it follows from (a) that

$$\left| \frac{f'(w)}{f(w)} \right| \leq \frac{4}{1 - |w|^2}.$$

Since this is true for every $w \in \mathbb{D}$, we can integrate the above inequality along the straight line segment from 0 to any $z \in \mathbb{D}$:

$$\int_0^z \left| \frac{f'(w)}{f(w)} \right| |dw| \leq 4 \int_0^z \frac{1}{1 - |w|^2} |dw| = 2 \log \left(\frac{1 + |z|}{1 - |z|} \right).$$

Therefore

$$\log \left| \frac{f(z)}{f(0)} \right| \leq \left| \log \left(\frac{f(z)}{f(0)} \right) \right| \leq \int_0^z \left| \frac{f'(w)}{f(w)} \right| |dw| \leq \log \left(\frac{1+|z|}{1-|z|} \right)^2,$$

which means

$$\left| \frac{f(z)}{f(0)} \right| \leq \left(\frac{1+|z|}{1-|z|} \right)^2.$$

Finally, the last inequality $|f(z)/f(0)| \geq (1-|z|)^2/(1+|z|)^2$ follows from the above argument applied to the univalent function $1/f$.

H. Entire Maps and Normal Families

H1. By Cauchy's Formula

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi r^n} \int_0^{2\pi} |f(re^{i\theta})| d\theta \leq \frac{n!}{2\pi} r^{\frac{17}{3}-n} \quad (0 < r < +\infty).$$

If $n \geq 6$, let $r \rightarrow +\infty$ to obtain $f^{(n)}(0) = 0$. If $n \leq 5$, let $r \rightarrow 0$ to obtain $f^{(n)}(0) = 0$. Since f is holomorphic, it follows that $f \equiv 0$. (Note that the crucial point of this argument is the fact that $17/3$ is not an integer.)

H2. Let γ be the square of side-length $2R > 0$ centered at the origin. By Cauchy's Formula, for every $n \geq 0$,

$$\begin{aligned} f^{(n)}(0) &= \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{z^{n+1}} dz \\ &= \frac{n!}{2\pi i} \left[\int_{-R}^R \frac{f(x-iR)}{(x-iR)^{n+1}} dx + \int_{-R}^R \frac{f(R+iy)}{(R+iy)^{n+1}} idy \right. \\ &\quad \left. - \int_{-R}^R \frac{f(x+iR)}{(x+iR)^{n+1}} dx - \int_{-R}^R \frac{f(-R+iy)}{(-R+iy)^{n+1}} idy \right]. \end{aligned}$$

It follows that

$$\begin{aligned} |f^{(n)}(0)| &\leq \frac{n!}{2\pi} \left[\int_{-R}^R \frac{|x|^{-\frac{1}{2}}}{R^{n+1}} dx + \int_{-R}^R \frac{R^{-\frac{1}{2}}}{R^{n+1}} dy + \int_{-R}^R \frac{|x|^{-\frac{1}{2}}}{R^{n+1}} dx + \int_{-R}^R \frac{R^{-\frac{1}{2}}}{R^{n+1}} dy \right] \\ &\leq \frac{n!}{\pi} \left[\frac{2}{R^{n+1}} \int_0^R x^{-\frac{1}{2}} dx + 2R^{-\frac{1}{2}-n} \right] \\ &\leq \frac{n!}{\pi} (6R^{-\frac{1}{2}-n}) \end{aligned}$$

The last term tends to zero as $R \rightarrow +\infty$, so $f \equiv 0$.

H3. Assuming the existence of such entire maps, we have

$$G(z) = \frac{1 - f(z)F(z)}{g(z)}$$

for all z . It follows that $1 - f(p)F(p) = 0$ if $g(p) = 0$, and the order of $1 - fF$ at p is at least equal to the order of g at p . Conversely, if we can choose F such that at every zero p of g , $1 - fF$ vanishes with an order at least equal to that of g , it easily follows that the function $(1 - fF)/g$ is entire.

So let k be the order of p as a root of g , i.e., $g^{(i)}(p) = 0$ for $0 \leq i \leq k - 1$. The k conditions $(1 - fF)^{(i)}(p) = 0$ for $0 \leq i \leq k - 1$ then determine the values

$$F(p) = \frac{1}{f(p)}, F'(p), \dots, F^{(k-1)}(p)$$

uniquely in terms of the successive derivatives of f at p . By the Theorem of Mittag-Leffler ([R2], Theorem 15.13), there exists an entire map F with these given local data at the zeros of g , and this completes the proof.

H4. First we show that the open set $\{z \in \mathbb{C} : |f(z)| > |g(z)|\}$ is non-empty. Otherwise, $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. If $g(p) = 0$ for some p , then $f(p) = 0$ and it is easy to see that the order of f at p must be at least the same as the order of g at p . It follows that f/g has removable singularities at all zeros of g , hence it is an entire function. Since $|f/g| \leq 1$, by Liouville's Theorem $f = cg$ for some constant c . Now the condition $f(0) = 0$ and $g(0) = 1$ implies $c = 0$. This is a contradiction since f is assumed to be non-constant.

Now both sets $\{z \in \mathbb{C} : |f(z)| > |g(z)|\}$ and $\{z \in \mathbb{C} : |f(z)| < |g(z)|\}$ are open and non-empty, so the common boundary $\{z \in \mathbb{C} : |f(z)| = |g(z)|\}$ must be non-empty and infinite.

H5. First solution. Let $f(z) := z - e^z$, which satisfies $f(z + 2\pi i) = f(z) + 2\pi i$ for all z . By Picard's Theorem, f takes on every value in \mathbb{C} infinitely often with at most one exception. But there cannot be any exceptional value at all, since if f omits a value v , then it has to omit every value $v + 2\pi in$ for $n \in \mathbb{Z}$. Hence, in particular, f takes on the value $v = 0$ infinitely often.

Second solution. The function $f(z) := ze^{-z}$ takes on the value 0 exactly once at $z = 0$, so by Picard's Theorem it must assume every other value infinitely often. In particular, it assumes the value $v = 1$ infinitely often.

H6. For $n \geq 1$ define $\zeta_n : U \rightarrow \mathbb{C}$ by

$$\zeta_n(z) := \frac{f_n(z) - A(z)}{f_n(z) - B(z)}.$$

Each $z \mapsto \zeta_n(z)$ is holomorphic since $f_n(z) \neq B(z)$. Also, $\zeta_n(z) \neq 0$ since $f_n(z) \neq A(z)$, and $\zeta_n(z) \neq 1$ since $A(z) \neq B(z)$. It follows from Montel's Theorem that $\{\zeta_n\}$ is a normal family. Hence $\{f_n = (\zeta_n B - A)/(\zeta_n - 1)\}$ must also be normal.

H7. Let $f \in \mathcal{F}$ have the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$. We compute

$$\begin{aligned} \int_{\mathbb{D}} |f(z)|^2 dx dy &= \int_0^1 \int_0^{2\pi} r |f(re^{i\theta})|^2 d\theta dr \\ &= 2\pi \int_0^1 \left(\sum_{n=0}^{\infty} |c_n|^2 r^{2n+1} \right) dr \quad (\text{by Parseval}) \\ &= 2\pi \sum_{n=0}^{\infty} |c_n|^2 \int_0^1 r^{2n+1} dr \\ &= \pi \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}. \end{aligned}$$

It follows from the definition of \mathcal{F} that

$$\sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1} \leq \frac{M}{\pi}. \quad (27)$$

Now Cauchy-Schwarz Inequality implies that for $z \in \mathbb{D}$,

$$|f(z)| \leq \sum_{n=0}^{\infty} \frac{|c_n|}{\sqrt{n+1}} |z|^n \sqrt{n+1} \leq \left(\sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |z|^{2n}(n+1) \right)^{\frac{1}{2}}.$$

A simple geometric series computation shows that

$$\sum_{n=0}^{\infty} |z|^{2n}(n+1) = \frac{1}{(1-|z|^2)^2}.$$

It follows from this and (27) that

$$|f(z)| \leq \sqrt{\frac{M}{\pi}} \frac{1}{1-|z|^2}$$

for $f \in \mathcal{F}$ and $z \in \mathbb{D}$. This shows that \mathcal{F} is locally uniformly bounded, hence normal by Montel's Theorem.

H8. The answer is negative. Let $r < 1$ and for $n \geq 0$ define A_n as the closed annulus $r2^{-n-1} \leq |z| \leq r2^{-n}$. Since 0 is an essential singularity for f , the Theorem of Casorati-Weierstrass shows that there exists an increasing sequence of integers $n_k \rightarrow +\infty$ and points $z_k \in A_{n_k}$ such that $|f(z_k)| > k$ for all k . Let $w_k := 2^{n_k} z_k$ so that $r/2 \leq |w_k| \leq r$ and $|f_{n_k}(w_k)| > k$.

Now assume $\{f_n\}$ is normal in U . Then the last inequality $|f_{n_k}(w_k)| > k$ shows that some subsequence of $\{f_{n_k}\}$, which we still denote by $\{f_{n_k}\}$, converges to ∞ uniformly on the annulus A_0 . We show that this implies $\lim_{z \rightarrow 0} f(z) = \infty$ meaning 0 is a pole of f . This contradiction proves that $\{f_n\}$ cannot be normal.

So let $M > 0$ and find $j > 0$ such that $|f_{n_k}(w)| > M$ if $k \geq j$ and $w \in A_0$. This means that $|f| > M$ on the union of annuli $A_{n_j} \cup A_{n_{j+1}} \cup \dots \cup A_{n_k}$ for every $k \geq j$. If A denotes the annulus $r2^{-n_k-1} \leq |z| \leq r2^{-n_j}$, then $|f| > M$ on the boundary of A . By applying the Maximum Principle to $1/f$, we conclude that $|f| > M$ throughout A . Letting $k \rightarrow \infty$, we obtain $|f(z)| > M$ when $0 < |z| < r2^{-n_j}$. Hence $\lim_{z \rightarrow 0} f(z) = \infty$.

H9. The entire map $z \mapsto e^{f(z)}$ cannot take values 0 or 1 since $e^{f(z)} = 1 - e^{g(z)}$, so it must be constant by Picard's Theorem. Hence f and g are both constants.

H10. The answer is negative. Let $f = u + iv$ and in the (u, v) -plane, look at the parabola $u = v^2$ which is an infinite set. If the image of f never intersects this locus, Picard's Theorem implies that f must be constant.

H11. Let $g(z) := \sum_{n=0}^{\infty} a_n z^n$, $f \in \mathcal{F}$ and $z \in \mathbb{D}$ with $|z| = r$. We have

$$|f(z)| \leq \sum_{n=0}^{\infty} |c_n| |z|^n \leq \sum_{n=0}^{\infty} a_n |z|^n = g(r).$$

By the Maximum Principle, $|f(z)| \leq g(r)$ if $|z| \leq r$. It follows that \mathcal{F} is locally uniformly bounded, hence normal by Montel's Theorem.

H12. Write

$$P(z) - Q(z) = e^{g(z)} - e^{f(z)} = e^{g(z)}(1 - e^{f(z)-g(z)}).$$

Since the entire function $1 - e^{f(z)-g(z)}$ never takes the value 1, either it is constant or it must take the value 0 infinitely often by Picard's Theorem. In the first case, $f - g \equiv \lambda$ for some $\lambda \in \mathbb{C}$. But $f(0) = g(0)$ by the assumption, so $\lambda = 0$ and $f \equiv g$. In the second case, the left-hand side $P - Q$ must vanish infinitely often. Since $P - Q$

is a polynomial, we must have $P \equiv Q$, hence $e^{f(z)-g(z)} \equiv 1$, hence $f - g \equiv 2\pi in$. Again, by the same reasoning, $n = 0$ and $f \equiv g$.

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