Math 704 Problem Set 9

due Monday 4/28/2025

Problem 1. Prove that every $f \in \mathcal{O}(\mathbb{D}^*)$ with a pole or essential singularity at 0 has arbitrarily large unramified disks. (Hint: Consider the function $g(z) = f(1/z)$ which is holomorphic in $\{z : |z| > 1\}$ and show that $zg'(z)$ cannot stay bounded as $z \to \infty$. Use Corollary 11.3.)

Problem 2. Verify that Picard's little theorem is equivalent to the statement that there are no non-constant entire functions f and g which satisfy the equation $e^f + e^g = 1$.

Problem 3. Suppose f is a periodic entire function in the sense that $f(z+\omega) = f(z)$ for some $\omega \neq 0$. Show that f has a fixed point.

Problem 4. Let f be an entire function such that $f \circ f$ has no fixed point (i.e., $f(f(z)) \neq z$ for all $z \in \mathbb{C}$). Prove that $f(z) = z + c$ for some $c \neq 0$. (Hint: Use Picard's little theorem to show that the entire function $(f(f(z)) - z)/(f(z) - z)$ is constant. Another application of the same theorem then shows that f' must be constant.)

Problem 5. Let f be a non-constant entire function which omits the value q, and P be a polynomial which is not identically q. Prove that the equation $f(z) = P(z)$ has infinitely many solutions.

Problem 6.

(i) Give an example of a family of holomorphic functions $\mathbb{C} \to \mathbb{C} \smallsetminus \{0\}$ that fails to be normal.

(ii) Let $f_1(z) = z + z^2$ and define $\{f_n\}$ inductively by $f_n = f_1 \circ f_{n-1}$ for $n \geq 2$. Show that $\{f_n\}$ is not normal in any neighborhood of 0. (Hint: Look at the sequence $\{f_n''(0)\}$.)