Math 704 Problem Set 8

due Monday 4/21/2025

Problem 1. Let $\triangle p_1 p_2 p_3$ denote the closed triangle (interior and boundary) with vertices p_1, p_2, p_3 labeled counterclockwise. Show that for any two triangles $\triangle p_1 p_2 p_3$ and $\triangle q_1 q_2 q_3$ there exists a unique homeomorphism $f : \triangle p_1 p_2 p_3 \to \triangle q_1 q_2 q_3$ with $f(p_i) = q_i$ for $i = 1, 2, 3$ which is a biholomorphism between the interiors.

Problem 2. Let $U \subset \mathbb{C}$ be a simply connected domain bounded by a Jordan curve and (a, b, c, d) be an ordered quadruple of points on ∂U chosen in counterclockwise direction. Show that there is a conformal map $f : U \to \mathbb{D}$ which sends (a, b, c, d) to the vertices of a rectangle inscribed in $\mathbb{D}$, and that f is unique up to a rotation of the disk about 0. (Hint: It suffices to show that any quadruple on $\mathbb{T}$ can be mapped by an element of $\mathrm{Aut}(\mathbb{D})$ to the vertices of a rectangle, unique up to a rotation. You may want to recall that two quadruples can be mapped to each other by a Möbius transformation iff they have the same cross-ratio.)

Problem 3. Let S be the unit square $\{x + iy \in \mathbb{C} : 0 < x, y < 1\}$, U be a bounded simply connected domain in $\mathbb{C}$, and $f : S \to U$ be a conformal map. For each $y \in (0, 1)$, let $L(y)$ denote the length of the curve $\gamma_y : (0, 1) \to \mathbb{C}$ defined by $\gamma_y(x) = f(x + iy)$. Use the length-area method to verify the following:

(i) $L(y)$ is finite and therefore γ_y lands on both ends for a.e. $y \in (0, 1)$.

(ii) The majority of the γ_y aren't too long: The measure of the set of $y \in (0, 1)$ for which $L(y) \le \sqrt{2\,\mathrm{area}(U)}$ is at least $1/2$.

Problem 4. Let $f : \mathbb{D} \to U$ be a conformal map. Suppose the radial limits $f^*(e^{i\alpha})$ and $f^*(e^{i\beta})$ exist and are equal for some $0 \le \alpha < \beta < 1$. Show that the domain bounded by the curves $r \mapsto f(re^{i\alpha})$ and $r \mapsto f(re^{i\beta})$ for $0 \le r \le 1$ cannot be contained in U.

Problem 5. Show that the function

$$f(z) = \exp\left(\frac{z+1}{z-1}\right)$$

is bounded and holomorphic in $\mathbb{D}$, and $f(z) \to 0$ as $z \to 1$ radially. However, for every $a \in \mathbb{D}$ there exists a sequence $z_n \to 1$ such that $f(z_n) \to a$.

Problem 6. Define $f, g \in \mathcal{O}(\mathbb{D})$ by

$$g(z) = \exp\left(\frac{1+z}{1-z}\right) \qquad \text{and} \qquad f(z) = (1-z)\exp(-g(z)).$$

Prove that the radial limit $f^*(e^{it}) = \lim_{r\to 1} f(re^{it})$ exists everywhere and defines a continuous function on $\mathbb{T}$. However, f is not even bounded in $\mathbb{D}$. Why doesn't this contradict the maximum principle?