Math 704 Problem Set 7 Solutions

Problem 1. Suppose $f \in \mathcal{O}(\mathbb{C})$ maps real numbers to real numbers and imaginary numbers to imaginary numbers. Prove that $f(-z) = -f(z)$ for all $z \in \mathbb{C}$.

Since f maps real numbers to real numbers, the entire function $z \mapsto \overline{f(\overline{z})}$ agrees with f on the real axis, hence everywhere by the identity theorem. Similarly, since f maps imaginary numbers to imaginary numbers, the entire function $z \mapsto -\overline{f(-\overline{z})}$ agrees with f on the imaginary axis, hence everywhere. Thus,

$$f(z) = \overline{f(\overline{z})} = -\overline{f(-\overline{z})} \qquad \text{for all } z \in \mathbb{C}.$$

It follows that f is an odd function:

$$f(-z) = \overline{f(-\overline{z})} = -f(z) \qquad \text{for all } z \in \mathbb{C}.$$

Problem 2. Suppose $f \in \mathcal{O}(\mathbb{C})$ takes real values on both the real and imaginary axes. Show that $f(z) = g(z^2)$ for some $g \in \mathcal{O}(\mathbb{C})$.

Since f takes real values on the real axis, the argument of the previous problem shows that $f(z) = \overline{f(\overline{z})}$ for all $z \in \mathbb{C}$. Since f takes real values on the imaginary axis, the entire function $z \mapsto \overline{f(-\overline{z})}$ agrees with f on the imaginary axis, hence everywhere, showing that $f(z) = \overline{f(-\overline{z})}$ for all $z \in \mathbb{C}$. Thus,

$$f(-z) = \overline{f(-\overline{z})} = f(z) \qquad \text{for all } z \in \mathbb{C},$$

that is, f is an even function. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, it follows that $a_n = 0$ for all odd n. Now the entire function g represented by the power series $\sum_{n=0}^{\infty} a_{2n} z^n$ satisfies the relation $g(z^2) = f(z)$ for all $z \in \mathbb{C}$.

Problem 3. Suppose $f \in \mathcal{O}(\mathbb{C})$ and $|f(z)| = 1$ whenever $|z| = 1$. Show that f is of the form $f(z) = \lambda z^n$, where $|\lambda| = 1$ and n is an integer ≥ 0.

Since f is not identically zero, the set E of zeros of f in the unit disk is finite (possibly empty). Let E^* be the image of E under the reflection $z \mapsto 1/\overline{z}$ across the unit circle. Since $|f(z)| = 1$ whenever $|z| = 1$, the Schwarz reflection principle shows that the function $F : \mathbb{C} \smallsetminus (E \cup E^*) \to \mathbb{C}$ defined by

$$F(z) = \begin{cases} f(z) & z \in \mathbb{D} \smallsetminus E \\ \dfrac{1}{\overline{f(1/\overline{z})}} & \mathbb{C} \smallsetminus (\mathbb{D} \cup E^*) \end{cases}$$

is holomorphic. We have $F = f$ in $\mathbb{D} \smallsetminus E$, hence in $\mathbb{C} \smallsetminus (E \cup E^*)$ by the identity theorem. Because f is entire, it follows that every point of $E \cup E^*$ is removable for F. Thus, $f(z) = 1/\overline{f(1/\overline{z})}$ in $\mathbb{C} \smallsetminus \mathbb{D}$, hence in $\mathbb{C}$. In particular, either $E = \emptyset$ or $E = \{0\}$ (a non-zero $p \in E$ would force $1/\overline{p} \in E^*$ to be a pole of f).

Now let $n = \operatorname{ord}(f, 0) \geq 0$. Then $g(z) = f(z)/z^n$ has a removable singularity at $z = 0$, so it extends to a non-vanishing holomorphic function in $\mathbb{D}$. Since $|g(z)| = 1$ for $|z| = 1$, the maximum principle applied to g and $1/g$ gives $|g| \leq 1$ and $1/|g| \leq 1$ in $\mathbb{D}$. Hence $|g| = 1$ in $\mathbb{D}$. By the open mapping theorem g must take a constant value λ with $|\lambda| = 1$. This proves $f(z) = \lambda z^n$, as required.

Problem 4. Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $|f(z)| \to 1$ as $|z| \to 1$. Show that f is a finite Blaschke product of the form

$$f(z) = \lambda \prod_{j=1}^{n} \Big(\frac{z - a_j}{1 - \overline{a_j}\, z} \Big),$$

where $|\lambda| = 1$ and $|a_j| < 1$ for all $1 \leq j \leq n$.

The argument is similar to problem 3 except that now poles are allowed. To show a slightly different viewpoint, instead of taking care to remove zeros and poles E and E^* as we did in problem 3, we stop worrying about them by working with holomorphic *maps* on the Riemann sphere. This makes the argument simpler and shorter.

We may assume f is not constant (otherwise $f = \lambda$ and the product over the empty set of zeros is 1). Since $|f(z)| \to 1$ as $|z| \to 1$, we can use the Schwarz reflection principle to extend f to a holomorphic map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which commutes with the reflection $z \mapsto 1/\overline{z}$, so $F(1/\overline{z}) = 1/\overline{F(z)}$. By Theorem 3.20, F is a rational function with the property $F(\mathbb{D}) = \mathbb{D}$ and $F(\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}) = \widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$. In particular, all zeros of F are in $\mathbb{D}$ and all poles of F are in $\widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$. If $a_1, \ldots, a_n$ are the zeros of F listed according to their orders, then $1/\overline{a_1}, \ldots, 1/\overline{a_n}$ are the poles of F listed according to their orders. The Blaschke product $B(z) = \prod_{j=1}^{n} (z - a_j)/(1 - \overline{a_j}\, z)$ has the same zeros and poles of the same orders as F. Hence F/B is a rational function without zeros or poles, and therefore is a constant λ. Since $|F| = |B| = 1$ on the unit circle, we must have $|\lambda| = 1$, as required.

Comment. For another argument avoiding Schwarz reflection, see Theorem 7.51. Also, problem 3 follows easily from problem 4. Make sure you convince yourself why.

Problem 5. Suppose $f \in \mathcal{O}(\mathbb{C}^*)$ has a simple pole at 0 and $f(\mathbb{T}) \subset \mathbb{R}$. Show that

$$f(z) = \frac{a}{z} + b + \overline{a}\, z$$

for some constants $a \in \mathbb{C}^*$ and $b \in \mathbb{R}$.

Since f maps the unit circle to the real line, it makes sense to consider Schwarz reflection using $z \mapsto 1/\overline{z}$ in the domain and $z \mapsto \overline{z}$ in the target. So, consider $g \in \mathcal{O}(\mathbb{C}^*)$ defined by $g(z) = \overline{f(1/\overline{z})}$. Then $g = f$ on $\mathbb{T}$, hence everywhere in $\mathbb{C}^*$. In other words, f has the symmetry $f(z) = \overline{f(1/\overline{z})}$ for all $z \neq 0$. Imposing this condition on the Laurent series

$f(z) = \sum_{n=-1}^{\infty} a_n\, z^n$ in $\mathbb{C}^*$ gives

$$\sum_{n=-1}^{\infty} a_n\, z^n = \sum_{n=-1}^{\infty} \overline{a_n}\, z^{-n} = \sum_{n=-\infty}^{1} \overline{a_{-n}}\, z^n.$$

Uniqueness of Laurent series coefficients then implies

$$a_0 = \overline{a_0}, \ a_1 = \overline{a_{-1}}, \text{ and } a_n = 0 \text{ if } |n| \geq 2,$$

which is equivalent to the claim.

Problem 6. What can you say about a bounded holomorphic function defined in the domain $\{z \in \mathbb{C} : |z - i| > 1/2\}$ which takes real values on the segment $[-1, 1]$?

We claim that any such function f must be constant. Consider the function $g(z) = \overline{f(\overline{z})}$ which is bounded and holomorphic in the reflected domain $\{z \in \mathbb{C} : |z + i| > 1/2\}$. Since f takes real values on the interval $[-1, 1]$, we see that $g = f$ on $[-1, 1]$. By the identity theorem, $g = f$ in their common domain $\{z \in \mathbb{C} : |z \pm i| > 1/2\}$. This shows that the function $F : \mathbb{C} \to \mathbb{C}$ given by

$$F(z) = \begin{cases} f(z) & \text{if } |z - i| > 1/2 \\ g(z) & \text{if } |z - i| \leq 1/2 \end{cases}$$

is well defined and holomorphic. Since f and g are bounded, so is F. By Liouville's theorem, F is constant. It follows that f must be constant.