Math 704 Problem Set 7

due Monday 3/31/2025

Problem 1. Suppose $f \in \mathcal{O}(\mathbb{C})$ maps real numbers to real numbers and imaginary numbers to imaginary numbers. Prove that $f(-z) = -f(z)$ for all $z \in \mathbb{C}$.

Problem 2. Suppose $f \in \mathcal{O}(\mathbb{C})$ takes real values on both the real and imaginary axes. Show that $f(z) = g(z^2)$ for some $g \in \mathcal{O}(\mathbb{C})$.

Problem 3. Suppose $f \in \mathcal{O}(\mathbb{C})$ and $|f(z)| = 1$ whenever $|z| = 1$. Show that f is of the form $f(z) = \lambda z^n$, where $|\lambda| = 1$ and n is an integer ≥ 0. (Hint: Use Schwarz reflection across the unit circle to show $f(1/\bar{z})) = 1/\overline{f(z)}$.)

Problem 4. Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $|f(z)| \to 1$ as $|z| \to 1$. Show that f is a finite Blaschke product of the form

$$f(z) = \lambda \prod_{j=1}^{n} \left(\frac{z - a_j}{1 - \overline{a_j}\, z} \right),$$

where $|\lambda| = 1$ and $|a_j| < 1$ for all $1 \leq j \leq n$.

Problem 5. Suppose $f \in \mathcal{O}(\mathbb{C}^*)$ has a simple pole at 0 and $f(\mathbb{T}) \subset \mathbb{R}$. Show that

$$f(z) = \frac{a}{z} + b + \overline{a}\, z$$

for some constants $a \in \mathbb{C}^*$ and $b \in \mathbb{R}$.

Problem 6. What can you say about a bounded holomorphic function defined in the domain $\{z \in \mathbb{C} : |z - i| > 1/2\}$ which takes real values on the segment $[-1, 1]$?