Math 704 Problem Set 6 Solutions

Problem 1. Suppose $f \in \mathcal{O}(\mathbb{D})$ and the sequence $\{f^{(n)}(0)\}_{n \geq 1}$ grows at most exponentially fast, i.e., there is a constant $\lambda > 1$ such that $|f^{(n)}(0)| < \lambda^n$ for all $n \geq 1$. Show that f extends to an entire function.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. By the assumption,

$$|a_n| = \frac{|f^{(n)}(0)|}{n!} \leq \frac{\lambda^n}{n!} \Longrightarrow 0 \leq |a_n|^{1/n} \leq \frac{\lambda}{(n!)^{1/n}}$$

for all n. Since $(n!)^{1/n}$ is easily seen to tend to infinity as $n \to \infty$, it follows that $\lim_{n\to\infty} |a_n|^{1/n} = 0$. Thus, the radius of convergence $R = 1/\lim_{n\to\infty} |a_n|^{1/n}$ of the power series of f is $+\infty$. As such, this power series provides an extension of f to an entire function.

Problem 2. Let f be a holomorphic function defined in a neighborhood of the origin, say $\mathbb{D}(0, r)$, which satisfies

$$f(2z) = (f(z))^2 \qquad \text{whenever } |z| < r.$$

Use this functional equation to show that f can be extended to an entire function. Can you determine all such entire functions explicitly?

Take $z \in \mathbb{C}$ and find the smallest integer $n \geq 0$ such that $|z|/2^n < r$. Set

$$F(z) = \left(f\left(\frac{z}{2^n}\right)\right)^{2^n}. \tag{1}$$

Observe that the right side of (1) remains unchanged if you replace n by any integer greater than n. In fact, if $|z|/2^k < r$, then $f(z/2^k) = f(2z/2^{k+1}) = (f(z/2^{k+1}))^2$ so $(f(z/2^k))^{2^k} = (f(z/2^{k+1}))^{2^{k+1}}$.

Thus, we have a well-defined function $F : \mathbb{C} \to \mathbb{C}$ which is holomorphic by (1) (if n works for some z, the same n works for all points sufficiently close to z). Moreover, $F(z) = f(z)$ for $|z| < r$ since we can take $n = 0$ in this case. Note that by (1) the functional equation $F(2z) = (F(z))^2$ still holds for all $z \in \mathbb{C}$. In particular, $F(0) = (F(0))^2$, so $F(0)$ is either 0 or 1.

To find all such F, we consider two cases:

Case 1. $F(0) = 0$. In this case F must be identically zero. Otherwise, let $m = \operatorname{ord}(F, 0) = \operatorname{ord}(f, 0) \geq 1$ and observe that in the equation $F(2z) = (F(z))^2$ the left side has a zero of order m at the origin while the right side has a zero of order $2m$ at the origin. Contradiction!

Case 2. $F(0) = 1$. In this case F has no zeros in $\mathbb{C}$. In fact, if $F(p) = 0$ for some p (necessarily $p \neq 0$), then $(F(p/2^n))^{2^n} = F(p) = 0$ so $F(p/2^n) = 0$ for all $n \geq 0$. By continuity, this implies $F(0) = 0$, which is a contradiction. Now F, being a non-vanishing entire function, must be of the form $F = \exp(G)$ for some $G \in \mathcal{O}(\mathbb{C})$ with $G(0) = 0$. Since

$$\exp(G(2z)) = F(2z) = (F(z))^2 = \exp(2G(z)) \qquad \text{for all } z \in \mathbb{C},$$

we have $G(2z) - 2G(z) = 2\pi i n$ for an integer n (independent of z). In view of $G(0) = 0$, we must have $n = 0$. Writing $G(z) = \sum a_k z^k$ and imposing the equation $G(2z) = 2G(z)$ then shows that $\sum a_k 2^k z^k = 2 \sum a_k z^k$ for all z, so $2^k a_k = 2a_k$ for all $k \geq 0$. This gives $a_k = 0$ for all $k \geq 0$ other than $k = 1$. Thus, G is linear of the form $G(z) = a_1 z$ and $F(z) = \exp(a_1 z)$ for an arbitrary $a_1 \in \mathbb{C}$.

Problem 3. The power series $f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots$ has radius of convergence 1, so $f \in \mathcal{O}(\mathbb{D})$. By Hadamard's gap theorem, $\mathbb{T}$ is the natural boundary of f. Verify this directly by showing that $\lim_{r \to 1} f(re^{2\pi i t}) = \infty$ for every dyadic rational t, i.e., those of the form $t = a/2^b$ for integers a, b.

First observe that $\lim_{r \to 1} f(r) = +\infty$. In fact, given any integer $N > 0$, since $\lim_{r \to 1} \sum_{n=0}^{N} r^{2^n} = N + 1$, we can find a sufficiently small $\delta > 0$ such that $1 - \delta < r < 1$ implies $f(r) > \sum_{n=0}^{N} r^{2^n} > N$. This can be interpreted as saying that the radial limit of f at 1 (the 2^0-th root of unity) is infinite.

Now the definition of f shows that $f(z^2) = z^2 + z^4 + z^8 + \cdots = f(z) - z$, or $f(z) = z + f(z^2)$. Using this relation, we see that if the radial limit of f at the 2^n-th roots of unity is infinite, then the radial limit of f at the 2^{n+1}-st roots of unity is also infinite. It follows inductively that the radial limit of f at every $z \in \mathbb{T}$ for which $z^{2^n} = 1$ for some $n \geq 0$ must be infinite. All such points are singular points of f and they form a dense subset of $\mathbb{T}$. Since the singular set is closed, it follows that every point of $\mathbb{T}$ is singular, i.e., $\mathbb{T}$ is the natural boundary of f.

Problem 4. Fix $\alpha > 0$ and let $f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$. Show that

(i) The power series has radius of convergence 1, so by Hadamard's gap theorem, $\mathbb{T}$ is the natural boundary of $f \in \mathcal{O}(\mathbb{D})$.

We have $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where $a_k = 2^{-n\alpha}$ if $k = 2^n$ for some $n \geq 0$ and $a_k = 0$ otherwise. Hence,

$$\limsup_{k \to \infty} |a_k|^{1/k} = \limsup_{n \to \infty} (2^{-n\alpha})^{2^{-n}} = \limsup_{n \to \infty} 2^{-n2^{-n}\alpha} = 2^0 = 1,$$

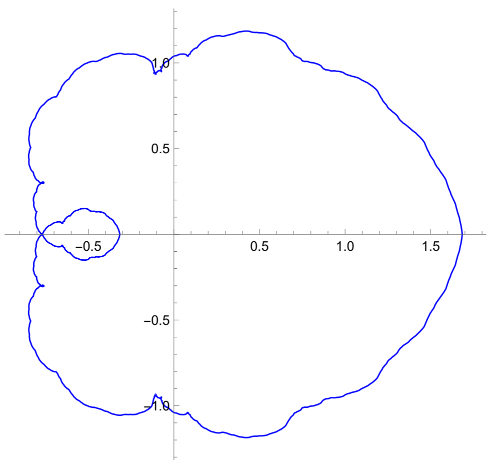
where we have used the fact that $\lim_{n \to \infty} n2^{-n} = 0$. It follows that the radius of convergence of the power series is 1. By Hadamard's gap theorem (Theorem 10.9 with $m_n = 2^n$), $\mathbb{T}$ is the natural boundary of f.

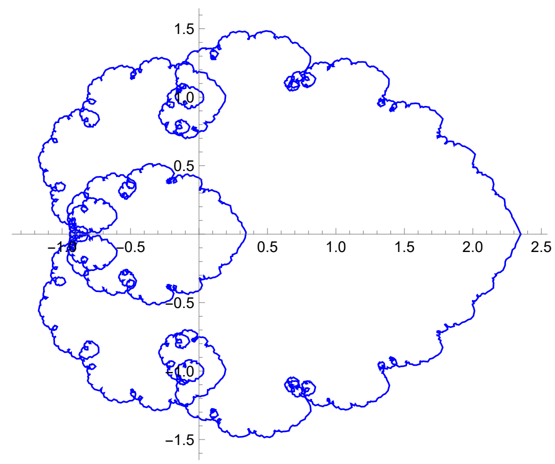
(ii) f has a continuous extension to the closed unit disk $\overline{\mathbb{D}}$. Moreover, if $\alpha > 1$ then $f|_{\mathbb{T}}$ is differentiable.

Since $|2^{-n\alpha} z^{2^n}| \leq 2^{-n\alpha}$ for $|z| \leq 1$ and $\sum 2^{-n\alpha}$ converges, the Weierstrass M-test shows that the power series converges uniformly on $\overline{\mathbb{D}}$. In particular, it defines a continuous function on $\overline{\mathbb{D}}$.

The power series representation $f'(z) = \sum_{n=0}^{\infty} 2^{n(1-\alpha)} z^{2^n-1}$ is valid in $\mathbb{D}$. We have $|2^{n(1-\alpha)} z^{2^n-1}| \leq 2^{n(1-\alpha)}$ for $|z| \leq 1$ and if $\alpha > 1$, $\sum 2^{n(1-\alpha)}$ converges. Hence by the Weierstrass M-test the power series of f' converges uniformly on $\overline{\mathbb{D}}$. This proves that if $\alpha > 1$ the restriction $f|_{\mathbb{T}}$ is differentiable and its derivative is $f'|_{\mathbb{T}}$.

Comment. It can be shown that when $0 < \alpha \leq 1$ the restriction $f|_{\mathbb{T}}$ is a nowhere differentiable curve. Compare the following graphs which render close approximations to this curve for $\alpha = 1.3$ (left) and $\alpha = 0.8$ (right):





Problem 5. Imitate the proof of Theorem 10.5 to show that every closed subset of $\mathbb{T}$ is the singular set of some holomorphic function in $\mathbb{D}$.

Let E be a non-empty closed subset of $\mathbb{T}$. If $E = \{q_1, \ldots, q_k\}$ is finite, the function $\sum_{n=1}^{k} 1/(z - q_n) \in \mathcal{O}(\mathbb{D})$ has the singular set E. So let us assume E is infinite. Take a dense sequence $\{q_n\}_{n\geq 1}$ in E, making sure that each isolated point of E (if any) appears infinitely often in this sequence. This is possible because E has at most countably many isolated points. For each n take a point $p_n \in \mathbb{C} \smallsetminus E$ such that $|p_n - q_n| < 1/n$. Since E is closed, the accumulation points of the sequence $\{p_n\}_{n\geq 1}$ all belong to E. We claim that in fact every $q \in E$ is an accumulation point of $\{p_n\}$. If q is not an isolated point of E, the density gives a subsequence $\{q_{n_j}\}$ converging to q. It follows from $|p_{n_j} - q| \leq |p_{n_j} - q_{n_j}| + |q_{n_j} - q| < 1/n_j + |q_{n_j} - q|$ that $p_{n_j} \to q$ as $j \to \infty$. On the other hand, if q is an isolated point of E, by the construction it appears infinitely often in the sequence $\{q_n\}$, so there is a subsequence $\{q_{n_j}\}$ taking the constant value q. It follows from $|p_{n_j} - q| = |p_{n_j} - q_{n_j}| < 1/n_j$ that $p_{n_j} \to q$ as $j \to \infty$. This proves the claim.

Now by the Weierstrass product theorem for general open sets (Theorem 8.25) there is an $f \in \mathcal{O}(\mathbb{C} \smallsetminus E)$ which vanishes precisely along $\{p_n\}$. The restriction of f to $\mathbb{D}$ has E as its singular set. Clearly every point of $\mathbb{T} \smallsetminus E \subset \mathbb{C} \smallsetminus E$ is a regular point of f. If some $q \in E$

were regular, we could extend f holomorphically to an open disk D centered at q. Then q would be an accumulation point of the zeros of f, so by the identity theorem $f = 0$ in D hence in $\mathbb{C} \setminus E$.

Problem 6. According to a theorem of Vivanti and Pringsheim (1893-1894), if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1 and $a_n \geq 0$ for all n, then $1 \in \mathbb{T}$ is a singular point of f. Prove this result by completing the following outline: Assume f extends holomorphically to a neighborhood of 1. Then the power series of f centered at $\frac{1}{2}$ would converge in the disk $\mathbb{D}(\frac{1}{2}, \frac{1}{2} + \varepsilon)$ for a small $\varepsilon > 0$. Hence $f(z) = \sum b_n (z - \frac{1}{2})^n$ for $|z - \frac{1}{2}| < \frac{1}{2} + \varepsilon$, where $b_n = \frac{1}{n!} f^{(n)}(\frac{1}{2})$ can be expressed as an infinite series involving the a_n. Substitute this expression for b_n and switch the order of summation to verify that $f(x) = \sum a_n x^n$ for real $1 < x < 1 + \varepsilon$, which would be a contradiction.

Let us follow the suggested outline. Assume by way of contradiction that 1 is regular and extend f holomorphically to an open disk B centered at 1. For small $\varepsilon > 0$ the disk $\mathbb{D}(\frac{1}{2}, \frac{1}{2} + \varepsilon)$ is contained in $\mathbb{D} \cup B$, so f has a power series representation of the form $f(z) = \sum_{n=0}^{\infty} b_n (z - \frac{1}{2})^n$ for $|z - \frac{1}{2}| < \frac{1}{2} + \varepsilon$. Here

$$\begin{aligned} b_n &= \frac{1}{n!} f^{(n)}\Big(\frac{1}{2}\Big) \\ &= \frac{1}{n!} \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\, a_k \Big(\frac{1}{2}\Big)^{k-n} \\ &= \frac{1}{n!} \sum_{k=n}^{\infty} \frac{k!}{(k-n)!}\, a_k \Big(\frac{1}{2}\Big)^{k-n} = \sum_{k=n}^{\infty} \binom{k}{n} a_k \Big(\frac{1}{2}\Big)^{k-n}. \end{aligned}$$

It follows that for real $1 < x < 1 + \varepsilon$,

$$f(x) = \sum_{n=0}^{\infty} b_n \Big(x - \frac{1}{2}\Big)^n = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \binom{k}{n} a_k \Big(\frac{1}{2}\Big)^{k-n} \Big(x - \frac{1}{2}\Big)^n.$$

Since all terms in this double series are positive (this is where we use the assumption $a_k \geq 0$), we can switch the order of summation to obtain

$$f(x) = \sum_{k=0}^{\infty} \Big[\underbrace{\sum_{n=0}^{k} \binom{k}{n} \Big(\frac{1}{2}\Big)^{k-n} \Big(x - \frac{1}{2}\Big)^n}_{\text{binomial expansion of } (x - \frac{1}{2} + \frac{1}{2})^k} \Big] a_k = \sum_{k=0}^{\infty} a_k x^k.$$

This is a contradiction because the power series $\sum a_k z^k$, having the radius of convergence 1, must diverge at every z with $|z| > 1$.