Math 704 Problem Set 6

due Monday 3/24/2025

Problem 1. Suppose $f \in \mathcal{O}(\mathbb{D})$ and the sequence $\{f^{(n)}(0)\}_{n\geq 1}$ grows at most exponentially fast, i.e., there is a constant $\lambda > 1$ such that $|f^{(n)}(0)| < \lambda^n$ for all $n \geq 1$. Show that f extends to an entire function.

Problem 2. Let f be a holomorphic function defined in a neighborhood of the origin, say $\mathbb{D}(0,r)$, which satisfies

$$f(2z) = (f(z))^2 \qquad \text{whenever } |z| < r.$$

Use this functional equation to show that f can be extended to an entire function. Can you determine all such entire functions explicitly? (Hint: For the latter question, study the cases $f(0) = 0$ and $f(0) = 1$ separately.)

Problem 3. The power series $f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots$ has radius of convergence 1, so $f \in \mathcal{O}(\mathbb{D})$. By Hadamard's gap theorem, $\mathbb{T}$ is the natural boundary of f. Verify this directly by showing that $\lim_{r\to 1} f(re^{2\pi i t}) = \infty$ for every dyadic rational t, i.e., those of the form $t = a/2^b$ for integers a, b. (Hint: Use $\lim_{r\to 1} f(r) = \infty$ together with the relation $f(z) = z + f(z^2)$.)

Problem 4. Fix $\alpha > 0$ and let $f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$. Show that

(i) The power series has radius of convergence 1, so by Hadamard's gap theorem, $\mathbb{T}$ is the natural boundary of $f \in \mathcal{O}(\mathbb{D})$.

(ii) f has a continuous extension to the closed unit disk $\overline{\mathbb{D}}$. Moreover, if $\alpha > 1$ then $f|_{\mathbb{T}}$ is differentiable.

Problem 5. Imitate the proof of Theorem 10.5 to show that every closed subset of $\mathbb{T}$ is the singular set of some holomorphic function in $\mathbb{D}$.

Problem 6. According to a theorem of Vivanti and Pringsheim (1893-1894), if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence 1 and $a_n \geq 0$ for all n, then $1 \in \mathbb{T}$ is a singular point of f. Prove this result by completing the following outline: Assume f extends holomorphically to a neighborhood of 1. Then the power series of f centered at $\frac{1}{2}$ would converge in the disk $\mathbb{D}(\frac{1}{2}, \frac{1}{2} + \varepsilon)$ for a small $\varepsilon > 0$. Hence $f(z) = \sum b_n (z - \frac{1}{2})^n$ for $|z - \frac{1}{2}| < \frac{1}{2} + \varepsilon$, where $b_n = \frac{1}{n!} f^{(n)}(\frac{1}{2})$ can be expressed as an infinite series involving the a_n. Substitute this expression for b_n and switch the order of summation to verify that $f(x) = \sum a_n x^n$ for real $1 < x < 1 + \varepsilon$, which would be a contradiction.