Math 704 Problem Set 5

due Monday 3/17/2025

Problem 1. Let K be a compact subset of the unit circle $\mathbb{T}$. If $K = \mathbb{T}$, every polynomial P satisfies $|P(0)| \leq \sup_{z \in K} |P(z)|$ by the maximum principle. If $K \neq \mathbb{T}$, this can fail dramatically. Show that in this case for every $\varepsilon > 0$ there is a polynomial P with $P(0) = 1$ such that $\sup_{z \in K} |P(z)| < \varepsilon$. (Hint: Use Runge's theorem on K for a suitable $f \in \mathcal{O}(\mathbb{C}^*)$.)

Problem 2. Show that there is a sequence $\{P_n\}$ of polynomials such that

$$\lim_{n \to \infty} P_n(z) = \begin{cases} 1 & \text{if } \mathrm{Im}(z) > 0 \\ 0 & \text{if } \mathrm{Im}(z) = 0 \\ -1 & \text{if } \mathrm{Im}(z) < 0. \end{cases}$$

Can you achieve the additional property

$$|P_n(z)| \leq 1 \qquad \text{for every } z \in \mathbb{D} \text{ and } n \geq 1?$$

Problem 3. Is there a sequence of polynomials which tends to 0 compactly in the upper half-plane but does not have a limit at any point of the lower half-plane?

Problem 4. Deduce Mittag-Leffler's Theorem 9.4 for an open set U from Runge's Theorem 9.18 by completing the following outline: Let $\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots$ be a nice exhaustion of U. For $n \geq 1$, let Q_n be the finite sum of the principal parts $P_k(1/(z - z_k))$ over all k such that $z_k \in K_n \smallsetminus K_{n-1}$. For each $n \geq 2$, find a rational function R_n with poles outside U such that $|Q_n - R_n| \leq 2^{-n}$ on K_{n-1}. Show that $f = Q_1 + \sum_{n=2}^{\infty}(Q_n - R_n)$ is a meromorphic function in U with the principal part $P_k(1/(z - z_k))$ at each z_k, and with no other poles.

Problem 5. Let $K \subset \mathbb{C}$ be compact and connected. Show that every connected component of $\widehat{\mathbb{C}} \smallsetminus K$ is simply connected.

Problem 6. Let $U, V \subset \mathbb{C}$ be simply connected domains with $U \cap V \neq \emptyset$. Show that every connected component of $U \cap V$ is simply connected. (Hint for both problems 5 and 6: Use the fact that simple connectivity of a domain Ω is equivalent to $\widehat{\mathbb{C}} \smallsetminus \Omega$ being connected, or to $H_1(\Omega) = 0$.)