Math 704 Problem Set 4 Solutions

Problem 1. Suppose $f \in \mathcal{M}(\mathbb{C}, \Lambda)$ has poles of order 2 along Λ and no other poles. Show that $f = a\wp + b$ for some constants a, b with $a \neq 0$.

We know that the sum of the residues of f over all poles in a fundamental parallelogram is zero (Theorem 9.8). Since the poles of f only occur along Λ, it follows that $\operatorname{res}(f, 0) = 0$. If we set $a = \lim_{z\to 0} z^2 f(z)$, it follows that f has the principal part a/z^2 at $z = 0$. By double periodicity, f has the principal part $a/(z-\omega)^2$ at every $\omega \in \Lambda$. This shows that the difference $f - a\wp$ has removable singularities at the points of Λ, so it extends to an entire function in $\mathcal{M}(\mathbb{C}, \Lambda)$. By Liouville's theorem for elliptic functions (Theorem 9.7), $f - a\wp$ is a constant b, as required.

Problem 2. Recall that E_2 is the Weierstrass elementary factor

$$E_2(z) = (1 - z)\exp\left(z + \frac{z^2}{2}\right).$$

(i) Show that the *Weierstrass σ-function* associated with Λ, defined by the infinite product

$$\sigma(z) = z \prod_{\omega\in\Lambda^*} E_2\left(\frac{z}{\omega}\right),$$

converges compactly in the plane, so $\sigma \in \mathcal{O}(\mathbb{C})$.

Recall the inequality

$$\left|E_2\left(\frac{z}{\omega}\right) - 1\right| \leq \left|\frac{z}{\omega}\right|^3$$

for $|z| < |\omega|$ (Lemma 8.20). Since $\sum_{\omega\in\Lambda^*} |\omega|^{-3} < +\infty$, the Weierstrass M-test shows that the series $\sum_{\omega\in\Lambda^*} |E_2(z/\omega) - 1|$ converges compactly in $\mathbb{C}$. It follows that the infinite product defining σ converges compactly in $\mathbb{C}$ as well, hence σ is an entire function.

(ii) Use logarithmic differentiation to show that $-(\sigma'/\sigma)' = \wp$.

Direct computation gives

$$\frac{E_2'(z)}{E_2(z)} = \frac{1}{z-1} + 1 + z.$$

Hence, by logarithmic differentiation,

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega\in\Lambda^*} \frac{1}{\omega}\frac{E_2'(z/\omega)}{E_2(z/\omega)} = \frac{1}{z} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right]$$

$$\Longrightarrow \left(\frac{\sigma'(z)}{\sigma(z)}\right)' = \frac{-1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{-1}{(z-\omega)^2} + \frac{1}{\omega^2}\right] = -\wp(z),$$

as claimed.

Problem 3. Consider the lattices $\Lambda = \langle \omega_1, \omega_2 \rangle = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ and $\Lambda' = \langle \omega_1', \omega_2' \rangle$, with $\operatorname{Im}(\omega_2/\omega_1) > 0$ and $\operatorname{Im}(\omega_2'/\omega_1') > 0$. Show that $\Lambda = \Lambda'$ if and only if

$$\begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix} \qquad \text{for some } a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1. \tag{1}$$

First suppose (1) holds. Then $\omega_2' = a\omega_2 + b\omega_1$ and $\omega_1' = c\omega_2 + d\omega_1$ both belong to Λ, so $\Lambda' \subset \Lambda$. Similarly, since

$$\begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix},$$

it follows that $\omega_2 = d\omega_2' - b\omega_1'$ and $\omega_1 = -c\omega_2' + a\omega_1'$ both belong to Λ', so $\Lambda \subset \Lambda'$. Thus, $\Lambda = \Lambda'$.

Conversely, if $\Lambda = \Lambda'$, then ω_1', ω_2' are integer linear combinations of ω_1, ω_2 and vice versa, so

$$\begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = A \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$$

for a 2×2 integer matrix A whose inverse A^{-1} is also an integer matrix. This implies $\det A = \pm 1$. The assumption that both ω_2/ω_1 and ω_2'/ω_1' have positive imaginary parts means that $\{\omega_1, \omega_2\}$ and $\{\omega_1', \omega_2'\}$ are positive bases for $\mathbb{R}^2$. Since A carries the first basis to the second, the linear map $\mathbf{x} \mapsto A\mathbf{x}$ of the plane must be orientation-preserving, so $\det A > 0$. Thus, $\det A = 1$ and (1) holds.

Problem 4. Show that there is a linear map $z \mapsto \alpha z$ carrying $\Lambda' = \langle 1, \tau' \rangle$ onto $\Lambda = \langle 1, \tau \rangle$ if and only if

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad \text{for some } a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1. \tag{2}$$

Prove that in this case

$$|\alpha|^2 = \frac{\operatorname{Im} \tau}{\operatorname{Im} \tau'} \tag{3}$$

and

$$\wp_{\Lambda'}(z) = \alpha^2 \wp_{\Lambda}(\alpha z). \tag{4}$$

If there is an $\alpha \in \mathbb{C}^*$ with $\alpha\Lambda' = \Lambda$, then $\langle \alpha, \alpha\tau' \rangle = \langle 1, \tau \rangle$. Hence, by problem 3,

$$\begin{bmatrix} \alpha\tau' \\ \alpha \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \tau \\ 1 \end{bmatrix} \tag{5}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. Dividing the pair of linear equations thus obtained gives (2). Conversely, if (2) holds, then (5) holds for the choice of

$$\alpha = c\tau + d \neq 0.$$

Using problem 3 again, we conclude that $\alpha\Lambda' = \Lambda$.

The computation

$$\begin{aligned} 2i \operatorname{Im}(\tau') = \tau' - \overline{\tau'} &= \frac{a\tau + b}{c\tau + d} - \frac{a\bar{\tau} + b}{c\bar{\tau} + d} \\ &= \frac{(ad - bc)(\tau - \bar{\tau})}{|c\tau + d|^2} \\ &= \frac{\tau - \bar{\tau}}{|\alpha|^2} = \frac{2i \operatorname{Im}(\tau)}{|\alpha|^2} \end{aligned}$$

gives (3). Finally, to prove (4), note that the function $f(z) = \alpha^2 \wp_\Lambda(\alpha z)$ is elliptic for the lattice $\alpha^{-1}\Lambda = \Lambda'$, and has a pole with the principal part

$$\frac{\alpha^2}{(\alpha z - \alpha\omega')^2} = \frac{1}{(z - \omega')^2}$$

at every $\omega' \in \Lambda'$. By problem 1, $f = \wp_{\Lambda'} + b$ for some $b \in \mathbb{C}$. Since the Laurent series of both f and $\wp_{\Lambda'}$ near 0 have zero constant terms, we must have $b = 0$.

Comment. The relation (3) can be interpreted geometrically as follows: The linear map $L : z \mapsto \alpha z$ induces an isomorphism $\mathbb{C}/\Lambda' \to \mathbb{C}/\Lambda$ between the quotient tori which changes the Euclidean area by a factor of $|\alpha|^2$ (the Jacobian of the map L). On the other hand, the areas of these tori coincide with the areas of their respective fundamental parallelograms, i.e., $\operatorname{Im}(\tau')$ and $\operatorname{Im}(\tau)$. Hence $\operatorname{Im}(\tau) = |\alpha|^2 \operatorname{Im}(\tau')$.

Problem 5. Think of the invariants g_2, g_3 of the lattice $\Lambda = \langle 1, \tau \rangle$ as functions of τ in the upper half-plane. Show that

$$\begin{aligned} g_2\left(\frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^4 \, g_2(\tau) \\ g_3\left(\frac{a\tau + b}{c\tau + d}\right) &= (c\tau + d)^6 \, g_3(\tau) \end{aligned}$$

whenever $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

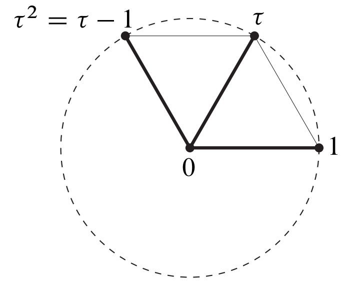
Let $\Lambda' = \langle 1, \tau' \rangle$, where $\tau' = (a\tau + b)/(c\tau + d)$. By problem 4, the non-zero complex number $\alpha = c\tau + d$ satisfies $\alpha\Lambda' = \Lambda$. It follows that

$$\begin{aligned} g_2(\tau') = 60 \sum_{\omega' \in \Lambda'^*} (\omega')^{-4} &= 60 \sum_{\omega \in \Lambda^*} (\omega/\alpha)^{-4} \\ &= \alpha^4 g_2(\tau) = (c\tau + d)^4 \, g_2(\tau), \end{aligned}$$

and similarly

$$g_3(\tau') = 140 \sum_{\omega' \in \Lambda'^*} (\omega')^{-6} = 140 \sum_{\omega \in \Lambda^*} (\omega/\alpha)^{-6}$$
$$= \alpha^6 g_3(\tau) = (c\tau + d)^6\, g_3(\tau).$$

Problem 6. Let $\tau = e^{i\pi/3}$. Show that the invariant $g_2(\tau)$ is zero.



It is easy to see (either algebraically or by drawing a picture) that $\tau^2 = \tau - 1$. Hence $\tau = (\tau - 1)/\tau$ and problem 5 with $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $\tau' = \alpha = \tau$ shows that

$$g_2(\tau) = \tau^4 g_2(\tau).$$

Since $\tau^4 \neq 1$, we conclude that $g_2(\tau) = 0$.