Math 704 Problem Set 4

due Monday 3/3/2025

In the following problems Λ is a given lattice in $\mathbb{C}$ and $\wp = \wp_\Lambda$ is the Weierstrass elliptic function associated with Λ.

Problem 1. Suppose $f \in \mathcal{M}(\mathbb{C}, \Lambda)$ has poles of order 2 along Λ and no other poles. Show that $f = a\wp + b$ for some constants a, b with $a \neq 0$. (Hint: Use $\operatorname{res}(f, 0) = 0$ to verify that the principal part of f at every $\omega \in \Lambda$ is $a/(z-\omega)^2$ for some $a \neq 0$ independent of ω.)

Problem 2. Recall that E_2 is the Weierstrass elementary factor

$$E_2(z) = (1-z)\exp\left(z + \frac{z^2}{2}\right).$$

(i) Show that the *Weierstrass σ-function* associated with Λ, defined by the infinite product

$$\sigma(z) = z \prod_{\omega \in \Lambda^*} E_2\left(\frac{z}{\omega}\right),$$

converges compactly in the plane, so $\sigma \in \mathcal{O}(\mathbb{C})$.

(ii) Use logarithmic differentiation to show that $-(\sigma'/\sigma)' = \wp$.

Problem 3. Consider the lattices $\Lambda = \langle \omega_1, \omega_2 \rangle = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ and $\Lambda' = \langle \omega_1', \omega_2' \rangle$, with $\operatorname{Im}(\omega_2/\omega_1) > 0$ and $\operatorname{Im}(\omega_2'/\omega_1') > 0$. Show that $\Lambda = \Lambda'$ if and only if

$$\begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$.

Problem 4. Show that there is a linear map $z \mapsto \alpha z$ carrying $\Lambda' = \langle 1, \tau' \rangle$ onto $\Lambda = \langle 1, \tau \rangle$ if and only if

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad \text{for some } a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1.$$

Prove that in this case

$$|\alpha|^2 = \frac{\operatorname{Im}\tau}{\operatorname{Im}\tau'}$$

and

$$\wp_{\Lambda'}(z) = \alpha^2 \wp_\Lambda(\alpha z).$$

(Hint: Assuming there is an $\alpha \neq 0$ with $\alpha\Lambda' = \Lambda$, use problem 3. For the last claim on the $\wp$-functions, use problem 1.)

Problem 5. Think of the invariants g_2, g_3 of the lattice $\Lambda = \langle 1, \tau \rangle$ as functions of τ in the upper half-plane. Show that

$$g_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^4\, g_2(\tau)$$
$$g_3\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^6\, g_3(\tau)$$

whenever $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. (Hint: Use problem 4 and the series definitions of g_2, g_3.)

Problem 6. Let $\tau = e^{i\pi/3}$. Show that the invariant $g_2(\tau)$ is zero.