Math 704 Problem Set 3 Solutions

Problem 1. In Example 9.2 we constructed a meromorphic function f in $\mathbb{C}$ with the principal part $1/(z-n)$ at every $n \in \mathbb{Z}$, and with no other poles:

$$f(z) = \frac{1}{z} + \sum_{n \in \mathbb{Z} \smallsetminus \{0\}} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

Show that in fact $f(z) = \pi \cot(\pi z)$.

For any $r > 0$ we showed that the tail $g(z) = \sum_{|n| \geq 2r} (1/(z-n) + 1/n)$ converges uniformly in $\mathbb{D}(0,r)$ and defines a holomorphic function there. It is therefore legitimate to find g' in $\mathbb{D}(0,r)$ by differentiating the series term-by-term. Since $f(z) = 1/z + \sum_{0<|n|<2r} (1/(z-n) + 1/n) + g(z)$, it follows that the formula

$$f'(z) = \frac{-1}{z^2} + \sum_{0<|n|<2r} \frac{-1}{(z-n)^2} + \sum_{|n| \geq 2r} \frac{-1}{(z-n)^2} = -\sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}$$

holds in $\mathbb{D}(0,r)$ away from the poles $z = n$ for which $|n| < r$. Since $r > 0$ was arbitrary, we conclude that $f'(z) = -\sum_{n \in \mathbb{Z}} 1/(z-n)^2$ for all $z \in \mathbb{C} \smallsetminus \mathbb{Z}$. But in problem 4 homework 1 we established the partial fraction expansion $\pi^2/\sin^2(\pi z) = \sum_{n \in \mathbb{Z}} 1/(z-n)^2$ in $\mathbb{C} \smallsetminus \mathbb{Z}$. This shows $f'(z) = -\pi^2/\sin^2(\pi z)$ in $\mathbb{C} \smallsetminus \mathbb{Z}$. Since $\mathbb{C} \smallsetminus \mathbb{Z}$ is connected, it follows that $f(z) = \pi \cot(\pi z) + C$ for some constant C.

To find C, we look at the Laurent series of each side near the origin. We have $f(z) = 1/z + \sum_{n \neq 0} (1/(z-n) + 1/n)$, where the series is holomorphic in a small neighborhood of the origin (in fact in $\mathbb{D}$) and takes the value zero there. Thus, $f(z) = 1/z + O(z)$ near the origin. On the other hand,

$$\begin{aligned}
\cot z = \frac{\cos z}{\sin z} &= \frac{1 - \frac{1}{2}z^2 + O(z^4)}{z - \frac{1}{6}z^3 + O(z^5)} = \frac{1}{z} \cdot \frac{1 - \frac{1}{2}z^2 + O(z^4)}{1 - \frac{1}{6}z^2 + O(z^4)} \\
&= \frac{1}{z} \left(1 - \tfrac{1}{2}z^2 + O(z^4)\right)\left(1 + \tfrac{1}{6}z^2 + O(z^4)\right) \\
&= \frac{1}{z} \left(1 - \tfrac{1}{3}z^2 + O(z^4)\right) = \frac{1}{z} - \tfrac{1}{3}z + O(z^3),
\end{aligned}$$

which shows $\pi \cot(\pi z) = 1/z - (\pi^2/3)z + O(z^3)$ near the origin. Since both the Laurent series of $f(z)$ and $\pi \cot(\pi z)$ are missing constant terms, it follows that $C = 0$.

Problem 2. Prove that

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z^3 - n^3}$$

defines a meromorphic function in $\mathbb{C}$. Identify the poles and principal parts of f.

As usual, convergence is shown using the Weierstrass M-test: Fix any $r > 0$ and note that if $|z| < r$ and $|n| \geq 2r$, then

$$|z^3 - n^3| \geq |n|^3 - |z|^3 \geq \frac{7}{8}\,|n|^3 \qquad \text{so} \qquad \frac{1}{|z^3 - n^3|} \leq \frac{8}{7}\,\frac{1}{|n|^3}.$$

Since $\sum_{n \neq 0} |n|^{-3} < +\infty$, the series $\sum_{|n| \geq 2r} 1/(z^3 - n^3)$ converges uniformly in $\mathbb{D}(0, r)$ to a holomorphic function g. Hence $f(z) = \sum_{|n|<2r} 1/(z^3 - n^3) + g(z)$ represents a meromorphic function in $\mathbb{D}(0, r)$. Since r was arbitrary, $f \in \mathcal{M}(\mathbb{C})$.

The poles of f are the roots of the equation $z^3 - n^3 = 0$ for integer n. They are of the form $n, n\lambda, n\lambda^2$, where $\lambda = e^{2\pi i/3}$. The principal part of f at $z = 0$ is clearly $1/z^3$. When $n \neq 0$, the identity

$$z^3 - n^3 = (z - n)(z - n\lambda)(z - n\lambda^2)$$

shows that the principal parts at $z = n$, $z = n\lambda$, and $z = n\lambda^2$ are

$$\begin{aligned}
\frac{1}{(n - n\lambda)(n - n\lambda^2)}\,\frac{1}{z - n} &= \frac{1}{3n^2}\,\frac{1}{z - n}, \\
\frac{1}{(n\lambda - n)(n\lambda - n\lambda^2)}\,\frac{1}{z - n\lambda} &= \frac{1}{3n^2\lambda^2}\,\frac{1}{z - n\lambda}, \text{ and} \\
\frac{1}{(n\lambda^2 - n)(n\lambda^2 - n\lambda)}\,\frac{1}{z - n\lambda^2} &= \frac{1}{3n^2\lambda}\,\frac{1}{z - n\lambda^2},
\end{aligned}$$

respectively.

Problem 3. Construct, using an explicit infinite series, a meromorphic function in $\mathbb{C}$ with the principal part $1/(z - \log n)$ at $\log n$ for every integer $n \geq 1$, and with no other poles.

For $n \geq 2$, let $Q_n(z)$ be the degree n Taylor polynomial of $1/(z - \log n)$ centered at 0:

$$Q_n(z) = -\sum_{k=0}^{n} \frac{z^k}{(\log n)^{k+1}}.$$

We claim that the series

$$f(z) = \frac{1}{z} + \sum_{n=2}^{\infty} \left[\frac{1}{z - \log n} - Q_n(z) \right]$$

defines a function with the desired property. To see this, fix any $r > 0$ and note that if $|z| < r$ and $n \geq e^{2r} \iff \log n \geq 2r$, then

$$\left|\frac{1}{z - \log n} - Q_n(z)\right| = \left|-\sum_{k=n+1}^{\infty} \frac{z^k}{(\log n)^{k+1}}\right| \leq \sum_{k=n+1}^{\infty} \frac{r^k}{(2r)^{k+1}}$$
$$= \frac{1}{r} \sum_{k=n+1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{r2^{n+1}}.$$

By the Weierstrass M-text, the series

$$\sum_{n \geq e^{2r}} \left[\frac{1}{z - \log n} - Q_n(z)\right]$$

converges uniformly in $\mathbb{D}(0, r)$ to a holomorphic function g. Hence

$$f(z) = \frac{1}{z} + \sum_{1<n<e^{2r}} \left[\frac{1}{z - \log n} - Q_n(z)\right] + g(z)$$

represents a meromorphic function in $\mathbb{D}(0, r)$ with the principal part $1/(z - \log n)$ at $z = \log n$ if $n < e^r$. Since r was arbitrary, $f \in \mathcal{M}(\mathbb{C})$ has the desired property.