Math 704 Problem Set 3

due Monday 2/24/2025

Problem 1. In Example 9.2 we constructed a meromorphic function f in $\mathbb{C}$ with the principal part $1/(z-n)$ at every $n \in \mathbb{Z}$, and with no other poles:

$$f(z) = \frac{1}{z} + \sum_{n \in \mathbb{Z} \smallsetminus \{0\}} \left(\frac{1}{z-n} + \frac{1}{n} \right).$$

Show that in fact $f(z) = \pi \cot(\pi z)$. (Hint: Use problem 4 homework 1 to verify that $f'(z) = -\pi^2 / \sin^2(\pi z)$, so $f(z) = \pi \cot(\pi z) + C$ for some constant C. Compare the Laurent expansions of both sides near the origin to deduce $C = 0$.)

Problem 2. Prove that

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z^3 - n^3}$$

defines a meromorphic function in $\mathbb{C}$. Identify the poles and principal parts of f. (Hint: For convergence, follow the usual M-test routine: Fix an arbitrary $r > 0$ and bound $1/|z^3 - n^3|$ from above for $|z| < r$ and $|n| \geq 2r$.)

Problem 3. Construct, using an explicit infinite series, a meromorphic function in $\mathbb{C}$ with the principal part $1/(z - \log n)$ at $\log n$ for every integer $n \geq 1$, and with no other poles. (Hint: Imitate the proof of Mittag-Leffler's Theorem 9.1. For $n \geq 2$ take $Q_n(z)$ to be the degree n Taylor polynomial of $1/(z - \log n)$ centered at 0.)