Math 704 Problem Set 2 Solutions

Problem 1.

(i) Construct an entire function with simple zeros at the points $\log n$ ($n \geq 1$), and with no other zeros.

Notice that the first zero is at $\log 1 = 0$. The Weierstrass product theorem (Theorem 8.21), with the choice $d_n = n$, shows that

$$f(z) = z \prod_{n=2}^{\infty} E_n\left(\frac{z}{\log n}\right)$$

is an entire function with the desired property. No canonical product with fixed $d_n = d$ would work because $\sum 1/(\log n)^{d+1}$ is divergent for every $d \geq 0$. However, the choice $d_n = n$ is not necessary, as something like $d_n = \lfloor \log n \rfloor$ would also work. In fact, you can easily check that this choice works for *any* Weierstrass product, no matter what the sequence of zeros might be.

(ii) Construct an entire function with a zero of order n at the point n ($n \geq 1$), and with no other zeros.

Looking at the condition of Theorem 8.21, it suffices to find an integer $d \geq 0$ such that the sum

$$\left(\frac{r}{1}\right)^{d+1} + 2\left(\frac{r}{2}\right)^{d+1} + 3\left(\frac{r}{3}\right)^{d+1} + \cdots = r^{d+1} \sum_{n=1}^{\infty} \frac{1}{n^d}$$

converges for any given $r > 0$. Clearly $d = 2$ will do, showing that

$$f(z) = \prod_{n=1}^{\infty} \left(E_2\left(\frac{z}{n}\right)\right)^n = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)^n \exp\left(z + \frac{z^2}{2n}\right)$$

is an entire funtion with the desired property.

Problem 2. Suppose $U \subset \mathbb{C}$ is a simply connected domain and $f \in \mathcal{O}(U)$ is not identically zero. Assume there is an integer $k \geq 2$ that divides the order of every zero of f. Show that f has a holomorphic k-th root in U, i.e., $f = g^k$ for some $g \in \mathcal{O}(U)$.

Arrange the distinct zeros of f in a (finite or infinite) sequence $\{z_n\}$ which has no accumulation point in U. Let $m_n = \operatorname{ord}(f, z_n)$. By the assumption, $\ell_n = m_n/k$ is an integer for all n. Use the Weierstrass product theorem for general domains (Theorem 8.25) to find an $h \in \mathcal{O}(U)$ with a zero of order ℓ_n at z_n for every n, and no other zeros. Then h^k vanishes precisely along $\{z_n\}$ and $\operatorname{ord}(h^k, z_n) = k \operatorname{ord}(h, z_n) = k\ell_n = m_n$. This shows that the ratio f/h^k has removable singularities at every z_n, so it extends to a non-vanishing holomorphic function ϕ in U. Since U is simply connected, ϕ has a holomorphic k-th root ψ in U. The product $g = h\psi \in \mathcal{O}(U)$ now satisfies $g^k = h^k \psi^k = h^k \phi = f$.

Problem 3. Let $f \in \mathcal{O}(\mathbb{C})$ and $M(r) = \sup_{|z|=r} |f(z)|$. Show that f is a polynomial if and only if

$$\limsup_{r\to+\infty} \frac{\log M(r)}{\log r} < +\infty. \tag{1}$$

If f is a polynomial of degree d, then $M(r) \leq \text{const.}\, r^d$ so $\log M(r) \leq \text{const.} + d \log r$ for all $r > 0$ and (1) clearly holds. Conversely, suppose (1) holds. Then there is a $d > 0$ such that $\log M(r)/\log r \leq d$ or $M(r) \leq r^d$ for all large r. By Cauchy's estimates,

$$|f^{(n)}(0)| \leq \frac{n!}{r^n} M(r) \leq n!\, r^{d-n}$$

for all large r. If $n > d$, it follows by letting $r \to \infty$ that $f^{(n)}(0) = 0$. This shows that

$$f(z) = \sum_{n\leq d} \frac{f^{(n)}(0)}{n!}\, z^n$$

is a polynomial of degree at most d.

Problem 4. Prove the following analog of Jensen's formula for meromorphic functions: Let f be meromorphic in $\mathbb{D}(0, R)$ with no zeros or poles at $z = 0$ or on the circle $|z| = r < R$. Let $z_1, z_2, \ldots, z_k$ and $p_1, p_2, \ldots, p_m$ denote the zeros and poles of f in $\mathbb{D}(0, r)$, each repeated as many times as its order. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{it})|\, dt = \log|f(0)| + \sum_{n=1}^{k} \log\left(\frac{r}{|z_n|}\right) - \sum_{n=1}^{m} \log\left(\frac{r}{|p_n|}\right). \tag{2}$$

We may assume $r = 1$; otherwise consider $f(rz)$ on the disk $\mathbb{D}(0, R/r)$ with zeros and poles at z_n/r and p_n/r. The finite Blaschke product

$$B(z) = \prod_{n=1}^{k} \left(\frac{z - z_n}{1 - \overline{z_n}\, z}\right) \cdot \prod_{n=1}^{m} \left(\frac{1 - \overline{p_n}\, z}{z - p_n}\right)$$

has zeros at the z_n and poles at the p_n, all of the same order as f. Hence $h = f/B$ has removable singularities and extends to a non-vanishing holomorphic function in $\mathbb{D}(0, 1 + \varepsilon)$ for some $\varepsilon > 0$, so $\log|h|$ is harmonic in $\mathbb{D}(0, 1 + \varepsilon)$. It follows that

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \log|f(e^{it})|\, dt &= \frac{1}{2\pi} \int_0^{2\pi} \log|h(e^{it})|\, dt \qquad \text{(since } |B(z)| = 1 \text{ for } |z| = 1) \\
&= \log|h(0)| \qquad \text{(by the mean value property of } \log|h|) \\
&= \log|f(0)| - \log|B(0)| \\
&= \log|f(0)| - \sum_{n=1}^{k} \log|z_n| + \sum_{n=1}^{m} \log|p_n|,
\end{aligned}$$

which is the formula (2) for $r = 1$.

Problem 5. Suppose f and g are bounded holomorphic functions in $\mathbb{D}$. If

$$f(e^{-1/n}) = g(e^{-1/n}) \qquad \text{for all } n \geq 1,$$

show that $f = g$ everywhere in $\mathbb{D}$.

The function $f - g \in \mathcal{O}(\mathbb{D})$ is bounded and vanishes along the sequence $z_n = e^{-1/n}$. Since

$$1 - |z_n| = 1 - e^{-1/n} = \frac{1}{n} + O\left(\frac{1}{n^2}\right) \qquad \text{as } n \to \infty,$$

the series $\sum_{n=1}^{\infty}(1 - |z_n|)$ diverges. It follows from Theorem 8.34 (see also Example 8.35) that $f - g$ is identically zero.

Problem 6. Let $\mathbb{H}$ denote the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $\{t_n\}$ be an increasing sequence of positive numbers with $\lim_{n\to\infty} t_n = +\infty$. Find a necessary and sufficient condition on $\{t_n\}$ for the existence of a bounded $f \in \mathcal{O}(\mathbb{H})$ with simple zeros along the sequence $\{i/t_n\}$. How would the answer change if we placed the zeros along the sequence $\{t_n + i\}$ instead?

We use the conformal isomorphism $\varphi : \mathbb{H} \to \mathbb{D}$ given by $w = \varphi(z) = (i - z)(i + z)$ to transfer the problem to the unit disk. Evidently $f \in \mathcal{O}(\mathbb{H})$ is bounded with simple zeros along $\{z_n\}$ if and only if $f \circ \varphi^{-1} \in \mathcal{O}(\mathbb{D})$ is bounded with simple zeros along $w_n = \varphi(z_n)$. It follows from Theorem 8.34 that a necessary and sufficient condition for the existence of such f is $\sum(1 - |w_n|) < +\infty$.

If $z_n = i/t_n$, then

$$w_n = \varphi\Big(\frac{i}{t_n}\Big) = \frac{i - \frac{i}{t_n}}{i + \frac{i}{t_n}} = \frac{t_n - 1}{t_n + 1}.$$

This gives

$$1 - |w_n| = 1 - w_n = \frac{2}{t_n + 1},$$

which is comparable to $1/t_n$ since $t_n \to +\infty$. Thus, the desired condition in this case is $\sum 1/t_n < +\infty$.

If, on the other hand, $z_n = t_n + i$, then

$$w_n = \varphi(t_n + i) = \frac{i - t_n - i}{i + t_n + i} = \frac{-t_n}{t_n + 2i}.$$

This gives

$$1 - |w_n| = 1 - \frac{t_n}{\sqrt{t_n^2 + 4}} = 1 - \frac{1}{\sqrt{1 + \frac{4}{t_n^2}}} = 1 - \Big(1 + \frac{4}{t_n^2}\Big)^{-1/2} = \frac{2}{t_n^2} + O\Big(\frac{1}{t_n^4}\Big),$$

where we have used the Taylor expansion $(1+x)^\alpha = 1 + \alpha x + O(x^2)$ near $x = 0$. Thus, $1 - |w_n|$ is comparable to $1/t_n^2$ and the desired condition in this case is $\sum 1/t_n^2 < +\infty$.

Comment 1. An alternative path to the second case is to first observe that since

$$1 \le \frac{1 - |w_n|^2}{1 - |w_n|} = 1 + |w_n| \le 2,$$

the terms $1 - |w_n|^2$ and $1 - |w_n|$ are comparable and we may as well look at the series $\sum(1 - |w_n|^2)$ (useful trick: It's often easier to work with absolute value squared). Now

$$1 - |w_n|^2 = 1 - \frac{t_n^2}{t_n^2 + 4} = \frac{4}{t_n^2 + 4},$$

which is comparable to $1/t_n^2$, giving the same condition as before.

Comment 2. The two cases of the problem contrast the difference between *radial* and *tangential* convergence to the boundary. In the first case $w_n \to 1$ along the real line while in the second case $w_n \to -1$ along a circle tangent to $\partial\mathbb{D}$ at -1.