Math 704 Problem Set 2

due Monday 2/17/2025

Problem 1.

(i) Construct an entire function with simple zeros at the points $\log n$ ($n \geq 1$), and with no other zeros.

(ii) Construct an entire function with a zero of order n at the point n ($n \geq 1$), and with no other zeros.

Problem 2. Suppose $U \subset \mathbb{C}$ is a simply connected domain and $f \in \mathcal{O}(U)$ is not identically zero. Assume there is an integer $k \geq 2$ that divides the order of every zero of f. Show that f has a holomorphic k-th root in U, i.e., $f = g^k$ for some $g \in \mathcal{O}(U)$. (Hint: Every non-vanishing holomorphic function in U has a holomorphic k-th root.)

Problem 3. Let $f \in \mathcal{O}(\mathbb{C})$ and $M(r) = \sup_{|z|=r} |f(z)|$. Show that f is a polynomial if and only if

$$\limsup_{r \to +\infty} \frac{\log M(r)}{\log r} < +\infty.$$

(Hint: For the "if" part use Cauchy estimates.)

Problem 4. Prove the following analog of Jensen's formula for meromorphic functions: Let f be meromorphic in $\mathbb{D}(0, R)$ with no zeros or poles at $z = 0$ or on the circle $|z| = r < R$. Let $z_1, z_2, \ldots, z_k$ and $p_1, p_2, \ldots, p_m$ denote the zeros and poles of f in $\mathbb{D}(0, r)$, each repeated as many times as its order. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{it})| \, dt = \log |f(0)| + \sum_{n=1}^{k} \log \left(\frac{r}{|z_n|} \right) - \sum_{n=1}^{m} \log \left(\frac{r}{|p_n|} \right).$$

(Hint: Normalize so $r = 1$; the Blaschke product

$$B(z) = \prod_{n=1}^{k} \left(\frac{z - z_n}{1 - \overline{z_n}\, z} \right) \cdot \prod_{n=1}^{m} \left(\frac{1 - \overline{p_n}\, z}{z - p_n} \right)$$

will help.)

Problem 5. Suppose f and g are bounded holomorphic functions in $\mathbb{D}$. If

$$f(e^{-1/n}) = g(e^{-1/n}) \qquad \text{for all } n \geq 1,$$

show that $f = g$ everywhere in $\mathbb{D}$.

Problem 6. Let $\mathbb{H}$ denote the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ and $\{t_n\}$ be an increasing sequence of positive numbers with $\lim_{n \to \infty} t_n = +\infty$. Find a necessary and sufficient condition on $\{t_n\}$ for the existence of a bounded $f \in \mathcal{O}(\mathbb{H})$ with simple zeros along the sequence $\{i/t_n\}$. How would the answer change if we placed the zeros along the sequence $\{t_n + i\}$ instead?