Math 704 Problem Set 1 Solutions

Problem 1. Prove the following form of Cauchy's criterion for convergence of infinite products: $\prod_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that $|\prod_{n=m}^{k} a_n - 1| < \varepsilon$ whenever $k \geq m \geq N$.

First assume $\prod_{n=1}^{\infty} a_n$ converges and let $\varepsilon > 0$ be arbitrary. By continuity of the function $(z, w) \mapsto z/w$ at $(1, 1)$ we can find a $0 < \delta < 1$ such that $|z - 1| < \delta$ and $|w - 1| < \delta$ imply $|z/w - 1| < \varepsilon$. By the convergence of $\prod_{n=1}^{\infty} a_n$ we can find $N \geq 1$ such that $|\prod_{n=m}^{\infty} a_n - 1| < \delta$ for all $m \geq N$ (in particular $a_n \neq 0$ for all $n \geq N$). Thus, if $k \geq m \geq N$,

$$\Big| \prod_{n=m}^{k} a_n - 1 \Big| = \Big| \frac{\prod_{n=m}^{\infty} a_n}{\prod_{n=k+1}^{\infty} a_n} - 1 \Big| < \varepsilon.$$

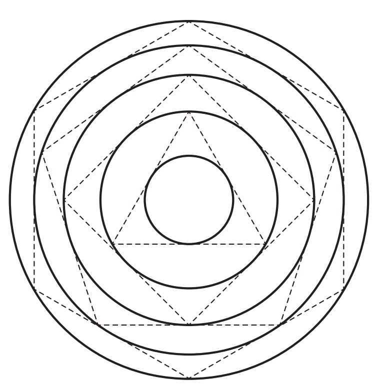
Now assume the infinite product satisfies the given Cauchy condition. Clearly $a_n \to 1$ as $n \to \infty$, so there is an n_0 such that $a_n \neq 0$ for all $n \geq n_0$. Take any $\varepsilon > 0$ and find $0 < \delta < 1$ such that $|z - 1| < \delta$ implies $|\operatorname{Log} z| < \varepsilon$. Find $N > n_0$ such that $|\prod_{n=m}^{k} a_n - 1| < \delta$ whenever $k \geq m \geq N$. Then

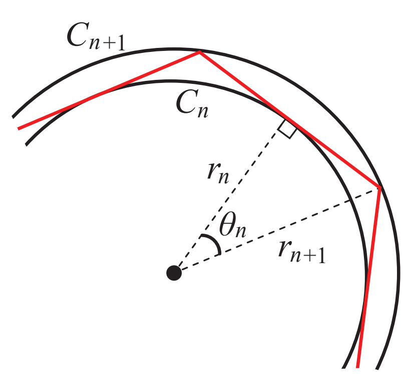
$$\Big| \sum_{n=m}^{k} \operatorname{Log} a_n \Big| = \Big| \operatorname{Log} \Big(\prod_{n=m}^{k} a_n \Big) \Big| < \varepsilon.$$

Since ε was arbitrary, this shows that $\{\sum_{n=n_0}^{k} \operatorname{Log} a_n\}$ is a Cauchy sequence. Thus, $\sum_{n=n_0}^{\infty} \operatorname{Log} a_n$ converges and it follows from Lemma 8.7 that $\prod_{n=1}^{\infty} a_n$ converges.

Question. Can you find a direct proof for the second implication that does not involve logarithms?

Problem 2. Consider a sequence $\{C_n\}_{n=1}^{\infty}$ of concentric circles of increasing radii. For each n, the circle C_n is inscribed in a regular $(n+2)$-gon and circumscribes a regular $(n+1)$-gon. Determine whether or not the radius of C_n tends to infinity as $n \to \infty$.





The regular $(n+2)$-gon is between C_n and C_{n+1}. If r_n denotes the radius of C_n, we have

$$\cos\theta_n = \frac{r_n}{r_{n+1}},$$

where $\theta_n = \pi/(n+2)$ (see the figure). Hence

$$r_{n+1} = r_n \sec\theta_n,$$

which gives the product formula

$$r_{n+1} = r_1 \prod_{k=1}^{n} \sec\theta_k.$$

Thus, to decide whether or not $r_n \to +\infty$, we need to determine the convergence or divergence of the infinite product $\prod_{k=1}^{\infty} \sec\theta_k$. As $x \to 0$,

$$\sec x - 1 = \frac{1}{\cos x} - 1 = \frac{1}{1 - x^2/2 + O(x^4)} - 1 = \frac{x^2}{2} + O(x^4),$$

which shows

$$|\sec\theta_k - 1| \le \text{const.}\ \theta_k^2 \le \frac{\text{const.}}{k^2}$$

for large k. Since $\sum_{k=1}^{\infty} 1/k^2 < +\infty$, we conclude that $\sum_{k=1}^{\infty} |\sec\theta_k - 1|$ converges. It follows that $\prod_{k=1}^{\infty} \sec\theta_k$ converges, so $\{r_n\}$ tends to a finite limit.

Problem 3. Let $f(z) = \prod_{n=0}^{\infty}(1 + z^{2^n})$.

(i) Show that the infinite product converges compactly in $\mathbb{D}$, so $f \in \mathcal{O}(\mathbb{D})$.

If $|z| \le r < 1$, then

$$|z|^{2^n} \le r^{2^n} \le r^n$$

for every $n \ge 0$. Since $\sum r^n$ converges, the Weierstrass M-test shows that $\sum_{n=0}^{\infty} z^{2^n}$ converges compactly in $\mathbb{D}$. Hence $\prod_{n=0}^{\infty}(1 + z^{2^n})$ converges compactly in $\mathbb{D}$.

(ii) Let $p_k(z) = \prod_{n=0}^{k}(1 + z^{2^n})$. Show that $p_k(z) = (1+z)p_{k-1}(z^2)$, and justify the functional equation $f(z) = (1+z)f(z^2)$.

We have

$$\begin{aligned} p_k(z) &= (1+z)(1+z^2)(1+z^4)\cdots(1+z^{2^k}) \\ &= (1+z)\prod_{n=0}^{k-1}(1 + (z^2)^{2^n}) = (1+z)p_{k-1}(z^2). \end{aligned}$$

Letting $k \to \infty$, we obtain $f(z) = (1+z)f(z^2)$ for every $z \in \mathbb{D}$.

(iii) Conclude that $f(z) = 1/(1-z)$.

Write $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $a_0 = f(0) = 1$ by the infinite product formula for f. By the functional equation in (ii),

$$\sum_{n=0}^{\infty} a_n z^n = (1+z) \sum_{n=0}^{\infty} a_n z^{2n} = \sum_{n=0}^{\infty} a_n z^{2n} + \sum_{n=0}^{\infty} a_n z^{2n+1}.$$

Comparing the coefficients of similar powers of z gives $a_n = a_{2n} = a_{2n+1}$ for all $n \geq 0$, so $a_n = a_0 = 1$ for all $n \geq 0$. Thus, $f(z) = \sum_{n=0}^{\infty} z^n = 1/(1-z)$.

(iv) What does the resulting identity

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = 1 + z + z^2 + z^3 + \cdots$$

for $|z| < 1$ tell you about the binary expansion of integers?

The right side contains every positive integer power of z once with coefficient 1, so the same must be true of the left side. It easily follows that every positive integer can be represented uniquely (up to permutation) as a sum of distinct non-negative powers of 2.

Problem 4. In this exercise you will prove *Euler's product formula* (1734):

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right) \qquad z \in \mathbb{C}.$$

The proof is deliberately divided into small steps for more clarity.

(i) Show that the above infinite product converges compactly in $\mathbb{C}$ to an entire function f with a simple zero at πn for every $n \in \mathbb{Z}$, and with no other zeros.

If $|z| \leq r$, then

$$\left|\frac{z^2}{\pi^2 n^2}\right| \leq \frac{r^2}{\pi^2 n^2}.$$

Since $\sum 1/n^2$ converges, the Weierstrass M-test shows that $\sum_{n=1}^{\infty} z^n/(\pi^2 n^2)$ converges compactly in $\mathbb{C}$. Hence, $\prod_{n=1}^{\infty}(1 - z^2/(\pi^2 n^2))$ converges compactly in $\mathbb{C}$ to an entire function. It follows that $f(z) = z \prod_{n=1}^{\infty}(1 - z^2/(\pi^2 n^2))$ is an entire function, with $f(z) = 0$ if and only if $z = 0$ or $1 - z^2/(\pi^2 n^2) = 0$ for some $n \geq 1$. Equivalently, $f(z) = 0$ if and only if $z = n\pi$ for some integer n. All zeros of f are simple because they occur as simple zeros of the factors z and $1 - z^2/(\pi^2 n^2) = (1 - z/(\pi n))(1 + z/(\pi n))$.

(ii) Show that $\sin z/f(z)$ has removable singularities at every πn, so it is an entire function with no zeros. Conclude that for some entire function g,

$$\sin z = e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right). \tag{1}$$

Since $\sin z$ also has simple zeros precisely at the integer multiples of π, the meromorphic function $\sin z/f(z)$ has removable singularities at the πn, so it extends to a non-vanishing entire function. Since $\mathbb{C}$ is simply connected, we can write this entire function as $\exp(g)$ for some $g \in \mathcal{O}(\mathbb{C})$ which is unique up to addition of an integer multiple of $2\pi i$. This proves (1).

(iii) Use logarithmic differentiation to show that

$$\cot z = g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z - \pi n} + \frac{1}{z + \pi n} \right), \tag{2}$$

and hence

$$g''(z) = -\frac{1}{\sin^2 z} + \sum_{n=-\infty}^{\infty} \frac{1}{(z - \pi n)^2}. \tag{3}$$

Show that the right hand side is invariant under the translation $z \mapsto z + \pi$, that is, $g''(z + \pi) = g''(z)$. Prove the estimate

$$|g''(z)| \le \frac{1}{\sinh^2 y} + 2 \sum_{n=0}^{\infty} \frac{1}{(\pi^2 n^2 + y^2)} \qquad (z = x + iy) \tag{4}$$

provided that $0 \le x \le \pi$. Use Liouville's theorem to conclude that $g'' = 0$ and hence g' is constant.

By logarithmic differentiation of (1),

$$\frac{(\sin z)'}{\sin z} = \frac{(e^{g(z)})'}{e^{g(z)}} + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(1 - z^2/(\pi^2 n^2))'}{1 - z^2/(\pi^2 n^2)},$$

so

$$\begin{aligned} \cot z &= g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z}{\pi^2 n^2 - z^2} \\ &= g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z - \pi n} + \frac{1}{z + \pi n} \right). \end{aligned}$$

More precisely, this means that the sequence

$$\cot z - \frac{1}{z} - \sum_{n=1}^{k} \left(\frac{1}{z - \pi n} + \frac{1}{z + \pi n} \right)$$

converges compactly in $\mathbb{C}$ to the entire function g' as $k \to \infty$ (the principal parts of $\cot z$ at $z = \pi n$ eventually cancel out with those of the infinite series). Differentiating term-by-term, which is legitimate under compact convergence, we obtain

$$g''(z) = (\cot z)' + \frac{1}{z^2} + \sum_{n=1}^{\infty} \left(\frac{1}{(z - \pi n)^2} + \frac{1}{(z + \pi n)^2} \right).$$

This is equivalent to (3). The function on the right side of (3) is invariant under the translation $z \mapsto z + \pi$. This is trivial for the infinite series and also holds for the term $-1/\sin^2 z$ because $\sin(z + \pi) = -\sin z$. Thus, g'' is π-periodic in the sense that $g''(z + \pi) = g''(z)$ for all $z \in \mathbb{C}$.

Note that if $z = x + iy$, then

$$|\sin z|^2 = \frac{1}{4}|e^{iz} - e^{-iz}|^2 \geq \frac{1}{4}\big||e^{iz}| - |e^{-iz}|\big|^2 = \frac{1}{4}|e^{-y} - e^{y}|^2 = \sinh^2 y.$$

Moreover, if $0 \leq x \leq \pi$, then $|z - \pi n|^2 = (x - \pi n)^2 + y^2$ is bounded below by $\pi^2 n^2 + y^2$ if $n \leq 0$ and by $\pi^2 (n-1)^2 + y^2$ if $n \geq 1$. Applying these estimates to the equation (3), we obtain (4).

Now $|g''|$ is trivially bounded on the closed rectangle $[0, \pi] \times [-1, 1]$. Since the right hand side of (4) is bounded above by $1/\sinh^2(1) + (2/\pi^2) \sum_{n=1}^{\infty} n^{-2}$ for $|y| \geq 1$, it follows that $|g''|$ is bounded on the strip $[0, \pi] \times (-\infty, +\infty)$ and hence on $\mathbb{C}$ by π-periodicity. Liouville's theorem then implies g'' is constant. Since $g''(x + iy) \to 0$ as $|y| \to +\infty$ by (4), we must have $g'' = 0$. Hence g' is constant.

(iv) The right hand side of (2) is an odd function. Show that this gives $g' = 0$, so g is constant. Use (ii) to conclude that $g = 0$.

The first claim is clear because $\cot z$ is odd. Since $1/z$ and the infinite series in (2) are also odd, the same must be true of g'. But the only constant odd function is zero, hence $g' = 0$ and g is constant. By (1),

$$e^{g(0)} = \lim_{z \to 0} \frac{\sin z}{z} = 1,$$

so $\exp(g(z)) = \exp(g(0)) = 1$ for all $z \in \mathbb{C}$. After adding an integer multiple of $2\pi i$, we may arrange $g = 0$.

Problem 5. Use the results of the previous problem to calculate

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Put $z = i\pi$ in Euler's product formula to get

$$\sin(i\pi) = i\pi \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right),$$

or

$$\frac{e^{-\pi} - e^{\pi}}{2i} = i\pi \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right),$$

or

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) = \frac{e^{\pi} - e^{-\pi}}{2\pi}.$$

To find the infinite sum, we use the equation (3) in the previous problem. Since

$$\sin^2 z = z^2 \left(1 - \frac{z^2}{6} + O(z^4)\right)^2 = z^2 \left(1 - \frac{z^2}{3} + O(z^4)\right)$$

as $z \to 0$, we have the Laurent series expansion

$$\frac{1}{\sin^2 z} = \frac{1}{z^2} + \frac{1}{3} + O(z^2)$$

near $z = 0$. It follows from (3) that

$$0 = g''(z) = -\frac{1}{3} + O(z^2) + \sum_{n=1}^{\infty} \left(\frac{1}{(z - \pi n)^2} + \frac{1}{(z + \pi n)^2}\right).$$

Letting $z \to 0$, we obtain

$$-\frac{1}{3} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = 0 \qquad \text{or} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Problem 6. Use the identity $\sin(2x) = 2 \sin x \, \cos x$ to show that

$$\prod_{n=2}^{\infty} \cos\left(\frac{\pi}{2^n}\right) = \frac{2}{\pi}.$$

Combine with the identity $\cos(2x) = 2\cos^2 x - 1$ to deduce *Veita's formula* (1579):

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots$$

We have

$$\begin{aligned} \sin(2x) &= 2 \sin x \cos x \\ &= 2^2 \sin(x/2) \cos(x/2) \cos x \\ &= 2^3 \sin(x/4) \cos(x/4) \cos(x/2) \cos x \\ &= \cdots \\ &= 2^{k+1} \sin(x/2^k) \prod_{n=0}^{k} \cos(x/2^n), \end{aligned}$$

so

$$\prod_{n=0}^{k} \cos(x/2^n) = \frac{\sin(2x)}{2^{k+1}\sin(x/2^k)}.$$

Substituting $x = \pi/4$ gives

$$\prod_{n=2}^{k+2} \cos(\pi/2^n) = \frac{1}{2^{k+1}\sin(\pi/2^{k+2})}.$$

Since $\sin(\pi/2^{k+2})/(\pi/2^{k+2}) \to 1$ or $2^{k+1}\sin(\pi/2^{k+2}) \to \pi/2$ as $k \to \infty$, it follows that

$$\prod_{n=2}^{\infty} \cos(\pi/2^n) = \frac{1}{\pi/2} = \frac{2}{\pi}.$$

If we start with the value $\cos(\pi/4) = \sqrt{2}/2$ and use the identity

$$\cos(x) = \sqrt{\frac{1+\cos(2x)}{2}},$$

we can inductively compute all the factors $\cos(\pi/2^n)$ in the above infinite product. For example,

$$\cos(\pi/8) = \sqrt{\frac{1+\cos(\pi/4)}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\cos(\pi/16) = \sqrt{\frac{1+\cos(\pi/8)}{2}} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$

and so on. Substituting these values into the above infinite product will give Veita's formula.