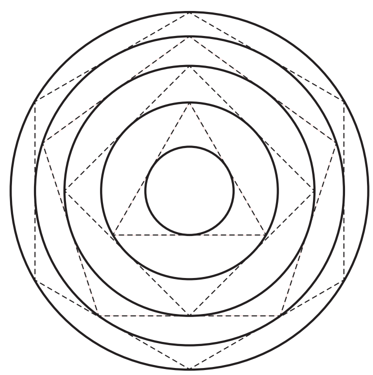
Math 704 Problem Set 1

due Friday 2/7/2025

Problem 1. Prove the following form of Cauchy's criterion for convergence of infinite products: $\prod_{n=1}^{\infty} a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that $|\prod_{n=m}^{k} a_n - 1| < \varepsilon$ whenever $k \geq m \geq N$. (Hint: The "only if" part is discussed in Remark 8.3. For the "if" part use logarithms.)

Problem 2. Consider a sequence $\{C_n\}_{n=1}^{\infty}$ of concentric circles of increasing radii. For each n, the circle C_n is inscribed in a regular $(n+2)$-gon and circumscribes a regular $(n+1)$-gon. Determine whether or not the radius of C_n tends to infinity as $n \to \infty$. (Hint: First verify the relation $r_{n+1} = r_n \sec \theta_n$, where r_n is the radius of C_n and $\theta_n = \pi/(n+2)$.)



Problem 3. Let $f(z) = \prod_{n=0}^{\infty}(1 + z^{2^n})$.

(i) Show that the infinite product converges compactly in $\mathbb{D}$, so $f \in \mathcal{O}(\mathbb{D})$.

(ii) Let $p_k(z) = \prod_{n=0}^{k}(1 + z^{2^n})$. Show that $p_k(z) = (1+z)p_{k-1}(z^2)$, and justify the functional equation $f(z) = (1+z)f(z^2)$.

(iii) Conclude that $f(z) = 1/(1-z)$.

(iv) What does the resulting identity

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = 1 + z + z^2 + z^3 + \cdots$$

for $|z| < 1$ tell you about the binary expansion of integers?

Problem 4. In this exercise you will prove *Euler's product formula* (1734):

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right) \qquad z \in \mathbb{C}.$$

The proof is deliberately divided into small steps for more clarity.

(i) Show that the above infinite product converges compactly in $\mathbb{C}$ to an entire function f with a simple zero at πn for every $n \in \mathbb{Z}$, and with no other zeros.

(ii) Show that $\sin z / f(z)$ has removable singularities at every πn, so it is an entire function with no zeros. Conclude that for some entire function g,

$$\sin z = e^{g(z)}\, z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right).$$

(iii) Use logarithmic differentiation to show that

$$\cot z = g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z - \pi n} + \frac{1}{z + \pi n}\right),$$

and hence

$$g''(z) = -\frac{1}{\sin^2 z} + \sum_{n=-\infty}^{\infty} \frac{1}{(z - \pi n)^2}.$$

Show that the right hand side is invariant under the translation $z \mapsto z + \pi$, that is, $g''(z + \pi) = g''(z)$. Prove the estimate

$$|g''(z)| \le \frac{1}{\sinh^2 y} + 2 \sum_{n=0}^{\infty} \frac{1}{(\pi^2 n^2 + y^2)} \qquad (z = x + iy)$$

provided that $0 \le x \le \pi$. Use Liouville's theorem to conclude that $g'' = 0$ and hence g' is constant.

(iv) The right hand side of the first identity in (iii) is an odd function. Show that this gives $g' = 0$, so g is constant. Use (ii) to conclude that $g = 0$.

Problem 5. Use the results of the previous problem to calculate

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Problem 6. Use the identity $\sin(2x) = 2 \sin x \cos x$ to show that

$$\prod_{n=2}^{\infty} \cos\left(\frac{\pi}{2^n}\right) = \frac{2}{\pi}.$$

Combine with the identity $\cos(2x) = 2\cos^2 x - 1$ to deduce *Veita's formula* (1579):

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} \cdot \frac{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}{2} \cdots$$