Math 704 Problem Set 10

due Monday 5/12/2025

Problem 1. Suppose U, V are domains in $\widehat{\mathbb{C}}$ and $f : U \to V$ is non-constant and holomorphic. Show that the following conditions are equivalent:

(i) f is a proper map: $f^{-1}(K) \subset U$ is compact whenever $K \subset V$ is compact.

(ii) For every sequence $\{z_n\}$ in U, if $z_n \to \partial U$ then $f(z_n) \to \partial V$.

(iii) f is a closed map: $f(E)$ is closed in V whenever E is closed in U.

(Hint for (iii) $\implies$ (i): Assuming $K \subset V$ is compact but $f^{-1}(K)$ is not, find a sequence $\{p_n\}$ in U with $p_n \to \partial U$ and $f(p_n) \to q \in K$ such that $f(p_n) \neq q$ for all n. Then use the assumption that f is closed to reach a contradiction.)

Problem 2. Suppose $f : \mathbb{D} \to \mathbb{D}$ is a proper holomorphic map of degree 2. Prove that there are $\phi, \psi \in \mathrm{Aut}(\mathbb{D})$ such that $(\psi \circ f \circ \phi)(z) = z^2$ for all $z \in \mathbb{D}$.

Problem 3. Suppose $P : \mathbb{C} \to \mathbb{C}$ is a monic polynomial of degree $d \geq 1$ such that $P^{-1}(\mathbb{D}) = \mathbb{D}$. Show that $P(z) = z^d$ for all $z \in \mathbb{C}$.

Problem 4. Suppose $f : \mathbb{D} \to \mathbb{D}$ is a proper holomorphic map such that $f(0) = 1/4$ and

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f'(z)}{f(z)}\, dz = 2 \qquad \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{z f'(z)}{f(z)}\, dz = 0 \qquad \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{z^2 f'(z)}{f(z)}\, dz = \frac{1}{2}.$$

Find a formula for $f(z)$. (Hint: The generalized argument principle is helpful here; see problem 27 of chapter 3.)

Problem 5. Prove that every proper holomorphic map between doubly connected domains (i.e., topological annuli) is a finite-degree covering map.

Problem 6. Suppose $U \subset \widehat{\mathbb{C}}$ is a finitely connected domain with $\chi(U) < 0$. Show that every proper holomorphic map $U \to U$ is a biholomorphism.