Math 363/663 Homework 9

due on Sunday 4/26/26

Problem 1. Find all harmonic functions in the unit disk $0 \leq r < 1, -\pi < \theta \leq \pi$ which depend only on r and not on the polar angle θ. How would the answer change if you replace the unit disk with the punctured unit disk $0 < r < 1, -\pi < \theta \leq \pi$?

Problem 2. Find the solution of the Laplace equation

$$\begin{cases} \Delta u = 0 & 0 \leq r < 1, -\pi < \theta \leq \pi \\ u(1,\theta) = \sin(3\theta) & -\pi < \theta \leq \pi \end{cases}$$

given by the Fourier method (separation of variables). Then compare your answer with the solution given by the Poisson integral formula to compute the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin(3t)}{5 - 4\sin t}\, dt.$$

Problem 3.

(i) Let $c > 0$ and consider the Poisson kernel

$$P(r,t) = \frac{c^2 - r^2}{c^2 - 2cr\cos t + r^2}$$

in the disk $0 \leq r < c$. Show that

$$\frac{c-r}{c+r} \leq P(r,t) \leq \frac{c+r}{c-r}.$$

(ii) Suppose u is a *positive* continuous function on the closed disk $0 \leq r \leq c$ which is harmonic inside $0 \leq r < c$. Use the Poisson integral formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)u(c,t)\, dt \qquad 0 \leq r < c, -\pi < \theta \leq \pi$$

to prove *Harnack's inequalities*

$$\frac{c-r}{c+r}\, u(0) \leq u(r,\theta) \leq \frac{c+r}{c-r}\, u(0) \qquad 0 \leq r < c, -\pi < \theta \leq \pi.$$

Here $u(0)$ denotes the value of u at the origin $r = 0$.

(iii) Use Harnack's inequalities to show that every positive harmonic function defined in the whole plane must be constant.