# Math 363/663 Homework 6

**due on Sunday 3/15/26**

**Problem 1.** Find the solution of the heat equation

$$\begin{cases} u_t = u_{xx} & 0 \le x \le 1,\ t \ge 0 \\ u(x,0) = 2x - x^2 & 0 \le x \le 1 \\ u(0,t) = 0, u_x(1,t) = 0 & t \ge 0. \end{cases}$$

Identify the steady-state and transient temperatures.

**Problem 2.** Show that the solution of the heat equation

$$\begin{cases} u_t = k u_{xx} & 0 \le x \le L,\ t \ge 0 \\ u(x,0) = f(x) & 0 \le x \le 1 \\ u(0,t) = \alpha(t),\ u(L,t) = \beta(t) & t \ge 0 \end{cases}$$

is unique by completing the following outline: Assume $u_1, u_2$ are two solutions of the above problem. Then $w = u_1 - u_2$ satisfies the heat equation but its initial and boundary conditions are identically zero. Consider the "energy integral"

$$E(t) = \int_0^L (w(x,t))^2 \, dx$$

which satisfies $E(0) = 0$ and $E(t) \ge 0$ for all $t \ge 0$. Show that $E$ is non-increasing by proving

$$E'(t) = -2k \int_0^L (w_x(x,t))^2 \, dx \le 0.$$

Conclude that $E$ and therefore $w$ must be identically zero.

**Problem 3.** Find the solution of the wave equation

$$\begin{cases} u_{tt} = u_{xx} & 0 \le x \le \pi,\ -\infty < t < +\infty \\ u(0,t) = u(\pi,t) = 0 & -\infty < t < +\infty \\ u(x,0) = \sin^2(x),\ u_t(x,0) = 0 & 0 \le x \le \pi. \end{cases}$$