

Math 363/663 Homework 4

due on Friday 2/27/26

Problem 1. Let $f(x) = \sin x$ for $0 \leq x \leq \pi$.

- (i) Is f a piecewise C^1 function on $[0, \pi]$? Sketch the graph of the π -periodic extension of f over a few periods.
- (ii) Compute the Fourier series of this extension. Use this Fourier series and a relevant convergence theorem to conclude that

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(2nx) \quad \text{if } 0 \leq x \leq \pi$$

- (iii) Evaluate both sides of the above formula at two suitable values of x to obtain the summation formulas

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} - \frac{\pi}{4}.$$

Problem 2. Let $f(x) = \pi - x$ for $0 < x < 2\pi$ and extend f as a 2π -periodic function on the real line. Sketch the graphs of $f(x)$, $f'(x)$ and $g(x) = \int_0^x f(t) dt$ for $-4\pi \leq x \leq 4\pi$. Without any computation, determine whether term-by-term differentiation and integration of the Fourier series of f give the Fourier series of f' and g .

Problem 3.

- (i) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic and

$$\text{FS}(f)(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)].$$

According to the basic differentiation theorem, when f is continuous and piecewise C^1 ,

$$\text{FS}(f')(x) = \sum_{n=1}^{\infty} [nb_n \cos(nx) - na_n \sin(nx)].$$

Can you explain the absence of the constant term in this Fourier series without using the basic differentiation theorem?

- (ii) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic and

$$g(x) = \int_0^x f(t) dt.$$

Show that g is 2π -periodic if and only if $\int_0^{2\pi} f(t) dt = 0$.