Math 310 Final Review Sheet
December 7, 2025

The final exam will be on Tuesday 12/16 from 11:00AM to 1:00PM in Kiely 320. It will have five problems, the first being a few multiple-choice questions. The test will be heavily biased towards the material we have studied since the last midterm, but clearly there are some basic ideas from earlier that you need to know. Learn the important definitions and theorems; you may be asked to state some. In addition to your lecture notes and relevant sections in the textbook, it would be a good idea to review the practice problems and solutions posted on the course webpage.

Essential ideas from earlier

definition of sup and inf; Heine-Borel: A subset of $\mathbb{R}$ is compact if and only if it is bounded and closed; limits of sequences; algebraic rules of limits; the squeeze theorem; every bounded monotone sequence is convergent; Bolzano-Weierstrass: every bounded sequence has a convergent subsequence; limit of a function at a point; sequential criterion for the existence of limit; definition of continuity of a function at a point; sums, products, quotients, and compositions of continuous functions are continuous; the extreme value theorem: a continuous function defined on a compact set assumes its maximum and minimum values; the intermediate value theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and $k \in \mathbb{R}$ is any number between $f(a)$ and $f(b)$, then there is a $c \in (a, b)$ with $f(c) = k$.

The derivative

definition of the derivative $f'(c)$; geometric interpretation of derivative as the slope of the tangent line; differentiability implies continuity; examples of differentiable and non-differentiable functions; algebraic rules of differentiation: linearity, product rule, quotient rule; the chain rule

Global results involving the derivative

Fermat: Suppose $f : (a, b) \to \mathbb{R}$ reaches a max/min at $c \in (a, b)$. If $f'(c)$ exists, then $f'(c) = 0$.

Rolle: Suppose f is continuous on $[a, b]$ and differentiable in (a, b). If $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

The mean value theorem (MVT): Suppose f is continuous on $[a, b]$ and differentiable in (a, b). Then

$$\frac{f(b) - f(a)}{b - a} = f'(c) \qquad \text{for some } c \in (a, b).$$

Corollaries of MVT: If $f' = 0$ everywhere in an interval I, then f is constant in I; if $f' > 0$ (resp. $f' < 0$) everywhere in I, then f is strictly increasing (resp. decreasing) in I

Cauchy's mean value theorem: Suppose f, g are continuous on $[a,b]$ and differentiable in (a,b). Then

$$(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c) \qquad \text{for some } c \in (a,b).$$

L'Hospital's rule and applications

Taylor's theorem (a generalization of MVT): Suppose f is $(n+1)$ times differentiable in an open interval I containing the point a. Then, for every $x \in I$,

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)\,(x-a)^k}_{P_n(x)} + \underbrace{\frac{1}{(n+1)!} f^{(n+1)}(c)\,(x-a)^{n+1}}_{R_n(x)}$$

for some c between x and a. Here $P_n(x)$ is called the nth Taylor polynomial of f at a. It is the unique polynomial of degree $\leq n$ whose first n derivatives at a match those of f. The term $R_n(x) = f(x) - P_n(x)$ is called the nth remainder or error.

Integration

Partitions, upper sum $U(f,P)$ and lower sum $L(f,P)$, the upper integral $U(f) = \inf_P U(f,P)$ and the lower integral $L(f) = \sup_P L(f,P)$; definition of the (Riemann) integral $\int_a^b f$; f is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

integration is a linear operation, has the additivity property, and preserves order; (piecewise) monotone functions are integrable; continuous functions are integrable

First Fundamental Theorem of Calculus (FTC1): Let f be integrable on $[a,b]$. Then $F(x) = \int_a^x f$ is continuous on $[a,b]$ with $F(a) = 0$. If f is continuous, then F is differentiable and $F' = f$.

Second Fundamental Theorem of Calculus (FTC2): Let f be integrable on $[a,b]$ and F be any antiderivative of f, i.e., any differentiable function such that $F' = f$. Then $\int_a^b f = F(b) - F(a)$.

Practice problems

1. True or false? Give a brief proof or a counterexample.

   (i) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and strictly increasing, then $f' > 0$ everywhere.

   (ii) If f is a quadratic polynomial, then $f = P_2$, where P_2 is the 2nd Taylor polynomial of f at any a.

   (iii) If $f + g$ is integrable on $[a,b]$, then both f and g are integrable on $[a,b]$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x^2 + x & \text{if } x > 0 \end{cases}$$

Differentiating either formula twice gives $f''(x) = 2$. Would it be correct then to say that $f''(x) = 2$ for all x? Why?

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable. If there are three points $a < b < c$ such that $f(a) = f(b) = f(c)$, show that f'' must vanish somewhere in (a, c).

4. Find the 6th Taylor polynomial $P_6(x)$ of $f(x) = \cos x$ at $a = 0$. Verify that $P_6(1)$ approximates $\cos(1)$ to within an error of less than 0.0002.

5. Discuss the integrability of the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A, B : \mathbb{R} \to \mathbb{R}$ are two differentiable functions. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \int_{A(x)}^{B(x)} f(t)\, dt.$$

Prove that g is differentiable and find a formula for $g'(x)$.

7. Prove the following mean value theorem for integrals: If f is continuous on $[a, b]$, then

$$\frac{1}{b-a} \int_a^b f = f(c) \qquad \text{for some } c \in (a, b).$$

Interpret the result geometrically. (Hint: Either use IVT, or combine MVT with FTC1.)