Math 320 Midterm 2 Review Sheet
November 5, 2025

The second midterm will be on Thursday 11/13 during your usual lecture time. It has four problems and is 90 minutes long. Here is a list of the topics that the exam is based on. Learn the important definitions; you may be asked to state some. In addition to your lecture notes and relevant sections in the textbook, it would be a good idea to review the practice problems and solutions posted on the course webpage.

Topology on $\mathbb{R}$

interior, closure and boundary of a set; open and closed sets; compact sets; Heine-Borel theorem: A subset of $\mathbb{R}$ is compact if and only if it is bounded and closed; Bolzano-Weierstrass theorem: Every bounded infinite subset of $\mathbb{R}$ has an accumulation point.

Sequences

definition of the limit of a sequence; convergent sequences are bounded; algebraic rules of limits; the squeeze theorem; monotone sequences: an increasing (resp. decreasing) sequence which is bounded above (resp. below) is convergent; definition of $\lim_{n\to\infty} x_n = +\infty$ or $-\infty$; Cauchy sequences; a sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence; subsequences; $\{x_n\}$ converges to L if and only if every subsequence of $\{x_n\}$ converges to L; every bounded sequence has a convergent subsequence (variant of the Bolzano-Weierstrass theorem).

Continuity

limit of a function at a point; sequential criterion for the existence of limit; algebraic rules of limits of functions; one-sided limits; ε-δ definition of continuity; two conditions equivalent to continuity of $f : D \to \mathbb{R}$ at $c \in D$: (i) for every sequence $\{x_n\}$ in D, if $x_n \to c$ then $f(x_n) \to f(c)$, and (ii) for every neighborhood V of $f(c)$ there is a neighborhood U of c such that $f(U \cap D) \subset V$; sums, products, quotients, and compositions of continuous functions are continuous.

Global properties of continuous functions

$f : D \to \mathbb{R}$ is continuous (everywhere) if and only if for every open set B there is an open set A such that $f^{-1}(B) = A \cap D$; in particular, if the domain D itself is open, $f : D \to \mathbb{R}$ is continuous if and only if for every open set B the preimage $f^{-1}(B)$ is open; if $f : D \to \mathbb{R}$ is continuous and $K \subset D$ is compact, then $f(K)$ is compact; the extreme value theorem: a continuous function defined on a compact set assumes its maximum and minimum values; the intermediate value theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and $k \in \mathbb{R}$ is any number between $f(a)$ and $f(b)$, then there is a $c \in (a, b)$ with $f(c) = k$.

Practice problems

1. True or false? Give a brief proof or a counterexample.

   (i) If the sequences $\{x_n\}$ and $\{x_n y_n\}$ are convergent, so is $\{y_n\}$.

   (ii) If $f : D \to \mathbb{R}$ is a continuous function, so is $|f| : D \to \mathbb{R}$ (as usual, $|f|$ denotes the function which takes the value $|f(x)|$ at each input x).

   (iii) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) \in \mathbb{Q}$ for every $x \in \mathbb{R}$, then f is a constant function.

2. Use the definition of limit to prove the following:

   (i) $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0.$

   (ii) $\displaystyle\lim_{x\to 2} (2x^2 + 3) = 11.$

3. Suppose we have a sequence $\{x_n\}$ of real numbers such that the subsequences $\{x_{2k}\}$ and $\{x_{2k-1}\}$ both converge to L as $k \to \infty$. Show that $\lim_{n\to\infty} x_n = L$.

4. Define a sequence $\{x_n\}$ by $x_1 = 0$ and $x_{n+1} = x_n/2 - 1$ for $n \geq 1$. Show that $\lim_{n\to\infty} x_n$ exists and find its value.

5. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x)| \leq \sqrt{|x|}$ for all $x \in \mathbb{R}$. Using the ε-δ definition of continuity, show that f is continuous at 0.

6. Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. Show that the range $f([a, b])$ is a closed interval.

7. Show that every continuous function $f : [0, 1] \to [0, 1]$ must have a *fixed point*, i.e., a point $c \in [0, 1]$ such that $f(c) = c$. Interpret this result geometrically. (Hint: Assuming $c = 0$ or $c = 1$ are not fixed points, look at the function $g(x) = f(x) - x$.)