Math 310 Midterm 1 Review Sheet

9/30/2025

The first midterm will be on Thursday 10/9 during your usual lecture time. It has four problems and is 90 minutes long. Here is a list of the topics that the exam is based on. Learn the important definitions; you may be asked to state some. In addition to your lecture notes and relevant sections in the textbook, it would be a good idea to review the practice problems and solutions posted on the course webpage.

Logic and techniques of proof

Quantifiers $\forall$ and $\exists$; quantified statements and their negation; proof by contradiction and by counterexample; mathematical induction

Set theory

Sets and subsets; equality of sets; the empty set; unions and intersections; the complement A^c and the difference $A \smallsetminus B = A \cap B^c$; De Morgan's laws; Cartesian products

Functions

Domain, co-domain and range; the image and preimage of a set under a function; surjective (=onto) functions; injective (=one-to-one) functions; bijective functions; the inverse function; composition of two functions

The real number system $\mathbb{R}$

$\mathbb{R}$ as an ordered field; the absolute value $|x|$ and its basic properties; supremum and infimum of a set; the completeness axiom; density of rationals and irrationals in $\mathbb{R}$: Every interval in $\mathbb{R}$ contains infinitely many rationals and infinitely many irrationals

Topology of $\mathbb{R}$

The r-neighborhood of x is $N(x,r) = \{t \in \mathbb{R} : |t - x| < r\} = (x - r, x + r)$.

For a set $S \subset \mathbb{R}$,

- $x \in S$ is an *interior point* of S if there is a neighborhood $N(x,r)$ such that $N(x,r) \subset S$. The set of all interior points of S is called the *interior* of S and is denoted by $\text{int}(S)$.
- $x \in \mathbb{R}$ is an *accumulation point* of S if every neighborhood $N(x,r)$ contains some point of S other than x. Equivalently, if every neighborhood $N(x,r)$ contains infinitely many points of S. The set of all accumulation points of S is called the *accumulation set* of S and is denoted by S'.
- the *closure* of S is the set $\text{cl}(S) = S \cup S'$. In other words,

$$\text{cl}(S) = \{x \in \mathbb{R} : \text{every neighborhood } N(x,r) \text{ contains some point of } S\}$$

- $x \in \mathbb{R}$ is a *boundary point* of S if every neighborhood $N(x, r)$ contains some point of S and some point of S^c. The set of all boundary points of S is called the *boundary* of S and is denoted by $\text{bd}(S)$.
- S is *open* if $S = \text{int}(S)$, i.e., if every point of S is an interior point of S.
- S is closed if $S = \text{cl}(S)$, i.e., if every accumulation point of S belongs to S.
- S is *compact* if every open cover of S has a finite subcover.

Moreover, we have the following facts:

- $\text{int}(S) \subset S \subset \text{cl}(S)$.
- $\text{cl}(S) = \text{int}(S) \cup \text{bd}(S)$.
- S is open $\iff S^c$ is closed.
- Arbitrary unions and finite intersections of open sets are open. Arbitrary intersections and finite unions of closed sets are closed.
- S is compact $\iff S$ is bounded and closed (Heine-Borel Theorem).

Practice Problems

1. Use proof by contradiction to show that (i) is true. Give a counterexample to show that (ii) is false.

(i) $\sqrt{2} + \sqrt{3} > \sqrt{8}$.

(ii) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} : |x| + |y| \leq 10$.

2. Guess a formula for the sum

$$1 + 3 + 5 + \cdots + (2n - 1)$$

of the first n odd numbers and prove it by induction.

3. Let $f : A \to B$ be a function. Show that for any subsets S, T of B, we have $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$. Show however that for $S, T \subset A$, the relation $f(S \cap T) = f(S) \cap f(T)$ may not be true. (Hint for the second part: Look at the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.)

4. Let A be the set of all continuous functions $f : [-1, 1] \to [0, 1]$. Consider the integration function $I : A \to \mathbb{R}$ defined by $I(f) = \int_0^1 f(x)\, dx$. Describe the range of I. Is I surjective? Is it injective?

5. Let $S \subset \mathbb{R}$ be non-empty and bounded. We know that in general $\sup(S)$ and $\inf(S)$ may not belong to S. Show however that $\sup(S)$ and $\inf(S)$ always belong to the closure $\text{cl}(S)$. Conclude that if S is non-empty and compact, $\sup(S)$ and $\inf(S)$ exist and both belong to S. In other words, *a compact subset of* $\mathbb{R}$ *has a maximum and a minimum element.*

6. For each set S, describe $\text{int}(S)$, $\text{cl}(S)$, and $\text{bd}(S)$. Determine if S is open, closed, compact, or none:

- $S = [-1, 0) \cup (0, 1]$
- $S = \{1/n : n \in \mathbb{N}\}$
- $S = \mathbb{N}$
- $S = \{x \in \mathbb{R} : |x - \sqrt{2}| \leq 1\}$
- $S = [0, 1] \cap \mathbb{Q}$