Math 310 Problem Set 11

12/2/2025

1. Let $f(x) = \sqrt{x}$.

(i) Find the 2nd Taylor polynomial P_2 of f at $a = 4$. Write your answer in powers of $x - 4$.

(ii) Use P_2 to approximate the value of $\sqrt{4.1}$ and estimate the error of this approximation.

2. Find the 8th Taylor polynomial P_8 of $f(x) = \sin x$ at $a = 0$. How accurate is the approximation of $\sin x$ by $P_8(x)$ on the interval $-2 \leq x \leq 2$?

3. True or false? Give a brief proof or a counterexample.

- If $f(x) = k$ for all $x \in [a, b]$ (a constant function), then f is integrable and $\int_a^b f(x)\, dx = k(b - a)$.
- If $|f|$ is integrable on $[a, b]$, so is f.
- If f, g are integrable on $[a, b]$ and h is any function that satisfies $f(x) \leq h(x) \leq g(x)$ for all $x \in [a, b]$, then h is also integrable on $[a, b]$.

4. Let $f(x) = x$. We know from calculus that $\int_a^b f(x)\, dx = (b^2 - a^2)/2$. In this exercise you will verify this result using the definition of integral. You may use the well-known sum

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

(i) Let $n \in \mathbb{N}$ and let P_n be the partition $\{x_0 = a, x_1, \ldots, x_n = b\}$ for which $\Delta x_i = x_i - x_{i-1} = (b - a)/n$ for every $1 \leq i \leq n$. Find

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \quad \text{and} \quad M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

(ii) Using (i), show that

$$L(f, P_n) = a(b - a) + \frac{n(n-1)}{2} \left(\frac{b - a}{n}\right)^2$$

and

$$U(f, P_n) = a(b - a) + \frac{n(n+1)}{2} \left(\frac{b - a}{n}\right)^2.$$

(iii) By taking the limit as $n \to \infty$ in (ii), show that

$$U(f) \leq \frac{b^2 - a^2}{2} \leq L(f).$$

Conclude that $\int_a^b f(x)\, dx$ exists and is equal to $(b^2 - a^2)/2$.

5. Suppose $f : [a, b] \to \mathbb{R}$ is a *continuous* function which satisfies $f(x) \geq 0$ for all $x \in [a, b]$. If $\int_a^b f(x)\,dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$. (Hint: Prove the contrapositive: Assume $f(x_0) > 0$ for some $x_0 \in [a, b]$ and show that $\int_a^b f(x)\,dx > 0$.)