Math 310 Problem Set 10

11/20/2025

1. Let f be differentiable and $f(0) = 1$. If $|f'(x)| \leq 1/2$ for all x, how large or small can $f(3)$ be?

2. Let f, g be differentiable on $[0, +\infty)$, with $f(0) = g(0)$. If $f'(x) \leq g'(x)$ for all $x > 0$, show that $f(x) \leq g(x)$ for all $x > 0$. (Physical interpretation: For two race cars starting at the same position, the car with the greater speed will always remain ahead.)

3. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$|f(x) - f(y)| \leq C|x - y|^{\alpha} \qquad \text{for all } x, y \in \mathbb{R}.$$

Here $C > 0$ and $\alpha > 1$ are fixed real numbers. Prove that f must be a constant function. (Hint: Show that $f'(x) = 0$ for all $x \in \mathbb{R}$.)

4. Let f be continuous on $[3, 5]$ and differentiable in $(3, 5)$. Suppose $f(3) = 6$ and $f(5) = 10$. Show that for some $c \in (3, 5)$ the tangent line to the graph of f at $(c, f(c))$ passes through the origin. (Hint: Consider $g(x) = f(x)/x$.)

5. Find the following limits:

- $\lim\limits_{x \to 0} \dfrac{\sin(x^2)}{1 - \cos x}$
- $\lim\limits_{x \to e} \dfrac{\log x - 1}{x}$
- $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^3}$
- $\lim\limits_{x \to 0} (1 + ax)^{1/x}$