Math 310 Problem Set 8

10/30/2025

1. Suppose f is continuous at 0 and $\lim_{x\to 0} \frac{f(x)}{x} = 2$. What is $f(0)$?

2. Give an example of

- a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set $A \subset \mathbb{R}$ such that the image $f(A)$ is not open.
- a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a compact set $K \subset \mathbb{R}$ such that the preimage $f^{-1}(K)$ is not compact.
- two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that g and $g \circ f$ are continuous, but f is not.
- a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous everywhere such that $|f|$ is continuous everywhere.

(Hint: Don't search for overly complicated examples. In the first three cases you can find very simple examples. For the last one, think of something like the Dirichlet function.)

3. Let $f : D \to \mathbb{R}$ be continuous at $c \in D$. If $f(c) > 0$, show that there is a $\delta > 0$ such that $|x - c| < \delta, x \in D$ implies $f(x) > 0$. Similarly, if $f(c) < 0$, there is a $\delta > 0$ such that if $|x - c| < \delta, x \in D$ implies $f(x) < 0$.

4. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous at $c = 0$ and discontinuous at every $c \neq 0$.

5. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions and $f(x) = g(x)$ for every $x \in \mathbb{Q}$. Show that $f(x) = g(x)$ for every $x \in \mathbb{R}$. In other words, *a continuous function on $\mathbb{R}$ is uniquely determined by its values at rational numbers*. What property of $\mathbb{Q}$ did you use in your proof?