Math 310 Problem Set 7

10/23/2025

1. Let a and b be positive numbers. Show that

$$\lim_{n\to\infty}(a^n + b^n)^{\frac{1}{n}} = \max\{a, b\}.$$

(Hint: Set $x_n = (a^n + b^n)^{1/n}$ and without loss of generality assume $a \leq b$. Verify that $b \leq x_n \leq 2^{1/n}\, b$ and apply the squeeze theorem.)

2. True for false? Give a brief proof or a counterexample.

- There is an unbounded sequence in $\mathbb{R}$ which has a subsequence converging to 0.
- There is a sequence in $\mathbb{R}$ which has three subsequences converging to -1, 0, and 5.
- The sequence $\{2^{\cos n}\}$ has a convergent subsequence.

3. Let S be a non-empty subset of $\mathbb{R}$.

(i) Show that $x \in \mathrm{cl}(S)$ if and only if there is a sequence $\{x_n\}$ in S such that $x_n \to x$.

(ii) Conclude that S is closed if and only if for every sequence $\{x_n\}$ in S with $x_n \to x$, the limit x is also in S.

4. Use the *definition* of limit to prove the following statements:

(i) $\lim\limits_{x\to 1}\,(x^2 - 4x) = -3$

(ii) $\lim\limits_{x\to 0}\, x\sin\left(\dfrac{1}{x}\right) = 0.$

(Hint for (ii): Use the fact that $|\sin\theta| \leq 1$ for all θ.)

5. Find the following limits or justify that the limit does not exist:

(i) $\lim\limits_{x\to 1}\, \dfrac{\sqrt{x} - 1}{x - 1}$

(ii) $\lim\limits_{x\to 0}\, \left(\dfrac{1}{x} - \dfrac{1}{x + 3x^2}\right)$

(iii) $\lim\limits_{x\to c}\, \lfloor x \rfloor$ (here $\lfloor x \rfloor$ denotes the integer part of the real number x).