Math 310 Problem Set 4

9/18/2025

Recap: For a given set $S \subset \mathbb{R}$,

- the *interior* of S is

$$\text{int}(S) = \{x \in S : \text{ some neighborhood of } x \text{ is contained in } S\}$$

- the *accumulation set* of S is

$$\begin{aligned} S' &= \{x \in \mathbb{R} : \text{every neighborhood of } x \text{ contains some point of } S \text{ other than } x\} \\ &= \{x \in \mathbb{R} : \text{every neighborhood of } x \text{ contains infinitely many points of } S\} \end{aligned}$$

- the *closure* of S is

$$\begin{aligned} \text{cl}(S) &= S \cup S' \\ &= \{x \in \mathbb{R} : \text{every neighborhood of } x \text{ contains some point of } S\} \end{aligned}$$

  Thus,

$$\text{int}(S) \subset S \subset \text{cl}(S).$$

- S is called *open* if $S = \text{int}(S)$.
- S is called closed if $S = \text{cl}(S)$
- S is *compact* if every open cover of S has a finite subcover.

  Moreover, we have the following facts:

- S is open $\iff S^c = \mathbb{R} \setminus S$ is closed.
- Arbitrary unions and finite intersections of open sets are open. Arbitrary intersections and finite unions of closed sets are closed.
- S is compact $\iff$ S is bounded and closed (Heine-Borel Theorem).

1. In each case, determine whether the set is open, closed, compact, or none:

(i) $[-1, 1] \cup \{2\}$

(ii) $\displaystyle\bigcup_{n=1}^{\infty} \left[\frac{1}{n+1}, \frac{1}{n}\right)$

(iii) $\{x \in \mathbb{R} : \sin x \geq 0\}$

(iv) $\{x \in (0, 1) : x \text{ is irrational}\}$

2. The *boundary* of a set $S \subset \mathbb{R}$ is defined as

$$\text{bd}(S) = \{x \in \mathbb{R} : \text{every neighborhood of } x \text{ contains points of both } S \text{ and } S^c\}.$$

For example, $\text{bd}([a, b]) = \text{bd}((a, b)) = \{a, b\}$ and $\text{bd}(\mathbb{Q}) = \mathbb{R}$. Find $\text{int}(S)$, S', $\text{cl}(S)$ and $\text{bd}(S)$ for each set S in problem 1.

3. Show that for every set $S \subset \mathbb{R}$,

$$\mathrm{cl}(S) = \mathrm{int}(S) \cup \mathrm{bd}(S).$$

(Hint: The inclusion $\mathrm{int}(S) \cup \mathrm{bd}(S) \subset \mathrm{cl}(S)$ is trivial, so you just need to prove $\mathrm{cl}(S) \subset \mathrm{int}(S) \cup \mathrm{bd}(S)$.)

4. True or false? Give a short proof or a counterexample.

(i) If S is not open, then S is closed.

(ii) If S is open and T is closed, then $S \setminus T$ is open.

(iii) If S and T are compact, so is $S \cup T$.

5. For any set $S \subset \mathbb{R}$, show that $\mathrm{int}(S)$ is open and $\mathrm{cl}(S)$ is closed by completing the following sketch:

To prove $\mathrm{int}(S)$ is open, we need to check that every $x \in \square$ is an interior point of $\mathrm{int}(S)$. We know there is a neighborhood $(x-r, x+r)$ contained in S. For every $y \in (x-r, x+r)$, let $c > 0$ be a number $\square$ than the distances between y and the points $x-r$ and $x+r$. Then $\square \subset (x-r, x+r) \subset S$, which shows $y \in \mathrm{int}(S)$. This proves $(x-r, x+r) \subset \square$, from which we conclude that x is an interior point of $\mathrm{int}(S)$.

To prove $\mathrm{cl}(S)$ is closed, we need to check that every accumulation point x of $\mathrm{cl}(S)$ belongs to $\square$. Take an arbitrary neighborhood $(x-r, x+r)$. We know that $(x-r, x+r)$ contains a point $y \in \square$ other than x. As before, we can find a small enough $c > 0$ such that $(y-c, y+c) \subset \square$. Since $y \in \mathrm{cl}(S)$, the neighborhood $(y-c, y+c)$ contains some point of $\square$. Thus, $(x-r, x+r) \cap S \neq \emptyset$. This proves that every neighborhood of x meets S, i.e., $x \in \mathrm{cl}(S)$.