# Math 310 Problem Set 3

**9/11/2025**

**1.** Let $\mathbb{F}$ be an ordered field. Use the axioms to verify the following statements:

(i) $1 > 0$.
(ii) $\forall\, x \in \mathbb{F},\ x > 0 \iff -x < 0$.
(iii) $\forall\, x \in \mathbb{F},\ x > 0 \iff 1/x > 0$.
(iv) $\forall\, x, y \in \mathbb{F},\ 0 < x < y \implies 0 < 1/y < 1/x$.
(v) $\forall\, x, y, z \in \mathbb{F},\ (x < y \text{ and } z < 0) \implies xz > yz$.
(vi) $\forall\, x \in \mathbb{F},\ x^2 + 1 > 0$.

(Those of you familiar with the field $\mathbb{C}$ of complex numbers will notice that (vi) shows $\mathbb{C}$ cannot be made into an ordered field, since $i^2 + 1 = 0$.)

**2.** Show that for all real numbers $x$ and $y$,

$$|\,|x| - |y|\,| \leq |x - y|.$$

This says that the distance between $|x|$ and $|y|$ is no more than the distance between $x$ and $y$. (Hint: You need to check the inequalities $-|x-y| \leq |x| - |y| \leq |x-y|$. Both follow from the triangle inequality $|a+b| \leq |a| + |b|$ for suitable choices of $a$ and $b$.)

**3.** In each case, find all the upper bounds (if any) and the least upper bound of $S \subset \mathbb{R}$:

- $S = \{a, b, c\}$, where $a > b > c$
- $S = \{n + (-1)^n : n \in \mathbb{N}\}$
- $S = \{x \in [0, \sqrt{10}] : x \text{ is rational}\}$
- $S = \{-\frac{3}{n} : n \in \mathbb{N}\}$

**4.** Are the following statements true or false? Justify your answers by a brief proof or counterexample.

- If $\sup(S) = \sup(T)$ and $\inf(S) = \inf(T)$, then $S = T$.
- If $a = \inf(S)$ and $a < a'$, then there is an $x \in S$ such that $a < x < a'$.
- If $b$ is an upper bound for $S$ and $b \in S$, then $b = \sup(S)$.
- If $S, T$ are bounded and $S \subset T$, then $\sup(S) \leq \sup(T)$ and $\inf(S) \geq \inf(T)$.

**5.** The field $\mathbb{R}$ of real numbers has the *Archimedean property (AP)*, which can be formulated as follows:

"For every real number $x > 0$, there exists an $n \in \mathbb{N}$ such that $n > x$."

This looks ridiculously obvious, but only because we are so used to our intuition of numbers that we just take it for granted. Amazingly, there are (incomplete) ordered fields in which this property fails. The following outline shows that (AP) is a consequence of the completeness axiom for $\mathbb{R}$. Fill in the blanks:

Suppose (AP) fails. Then we can find some ________ such that ________ for all $n \in \mathbb{N}$. In other words, the real number $x$ is an ________ for $\mathbb{N}$. By the ________, $\mathbb{N}$ must have the least upper bound $b \in \mathbb{R}$. Now the number $b - 1$ cannot be an upper bound for $\mathbb{N}$ because ________, so there must be an $n \in \mathbb{N}$ such that ________. But this implies ________ which contradicts the definition of $b = \sup(\mathbb{N})$. This contradiction shows that (AP) must hold.