Math 231 Midterm 2 Review Sheet, 4/11/2025

The second midterm will be on Thursday 4/24 during your usual lecture time. It has four problems (one being a few multiple-choice questions) and is 100 minutes long. You may use a calculator for basic arithmetic although it won't be something you need. *You are not allowed to use any built-in linear algebra packages on a calculator.* Here is a list of the topics that you are advised to study:

- Subspaces of \mathbb{R}^n ; the concepts of basis and dimension for a subspace; coordinates of a vector relative to a basis.
- Let *W* be a subspace of \mathbb{R}^n . Then the size of any linearly independent set in *W* is at most the size of any spanning set for *W*. In particular, if dim(*W*) = *k*, every linearly independent set in *W* contains *at most k* vectors while every spanning set for *W* contains *at least k* vectors.
- *Basis Test:* Consider *n* vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^n$ and put them as the columns of an $n \times n$ matrix *A*. Then,

 $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n \iff A$ is invertible $\iff \det(A) \neq 0$.

In this case, the coordinate vector of any $\mathbf{b} \in \mathbb{R}^n$ relative to $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is given by the solution of the system with the augmented matrix $[A|\mathbf{b}]$.

• The row space, column space and null space of a matrix; finding bases for these spaces by reducing to a row echelon form; definition of rank and nullity of a matrix; for any *m* × *n* matrix *A*,

 $0 \le \operatorname{rank}(A) \le \min\{m, n\}$ and $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$.

- Evaluating det(A) by the cofactor expansion formula along any row or column; determinant of a triangular matrix is the product of its main diagonal entries; det(A) = det(A^T); det(cA) = cⁿ det(A) if A is n × n; det(AB) = det(A) det(B); det(A⁻¹) = 1/det(A); Cramer's rule.
- Eigenvalues and eigenvectors of a square matrix; eigenvalues of a triangular matrix are its main diagonal entries; finding a basis for each eigenspace; algebraic vs. geometric multiplicity of an eigenvalue; if λ₁,..., λ_n are the eigenvalues of an n × n matrix A, then

$$det(A) = \lambda_1 \cdots \lambda_n$$
 and $tr(A) = \lambda_1 + \cdots + \lambda_n$.

• The concept of similar matrices; diagonalizable matrices; an $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors (special case: if there are n distinct eigenvalues); given a diagonalizable matrix A how to find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$; two applications:

• Finding the *k*-th power of *A* using the formula $A^k = PD^kP^{-1}$

• Suppose *A* is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then, for any vector \mathbf{v} ,

$$\mathbf{v} = c_1 \, \mathbf{v}_1 + \dots + c_n \, \mathbf{v}_n \Longrightarrow A^k \mathbf{v} = c_1 \, \lambda_1^k \, \mathbf{v}_1 + \dots + c_n \, \lambda_n^k \, \mathbf{v}_n.$$

- Useful characterizations of invertibility: The following conditions on an $n \times n$ matrix A are equivalent:
 - *A* is invertible
 - The equation $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$
 - The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$
 - The RREF of *A* is the identity matrix *I*
 - $det(A) \neq 0$
 - $\operatorname{rank}(A) = n$ and $\operatorname{nullity}(A) = 0$
 - The rows (or columns) of A are linearly independent
 - The rows (or columns) of *A* form a basis for \mathbb{R}^n
 - The eigenvalues of *A* are non-zero

Practice Problems.

1. Verify that $\mathcal{B} = \left\{ \begin{bmatrix} k \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ k \end{bmatrix} \right\}$ is a basis for \mathbb{R}^2 no matter what k is. Then set

- k = 1 and find the coordinate vector of $\begin{bmatrix} x \\ y \end{bmatrix}$ relative to \mathcal{B} .
- 2. Consider the matrix $A = \begin{bmatrix} 3 & -1 & 2 & 0 \\ 1 & 0 & 2 & -1 \\ 5 & -1 & 6 & -2 \end{bmatrix}$.
 - (i) Find bases for row(A) and col(A) and null(A).
 - (ii) What are rank(A) and nullity(A)?
- 3. For what values of *x* is the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ x & 3 & 2 \end{bmatrix}$ singular?

4. Let *A*, *B* be $n \times n$ matrices such that $B^{-1}AB = A^2$. What can you say about det(*A*)?

5. Solve the following system using Cramer's rule:

$$\begin{cases} 2x + 4y + 6z = 18\\ 4x + 5y + 6z = 24\\ 3x + y - 2z = 4 \end{cases}$$

6. Consider the matrix $A = \begin{bmatrix} 4 & -3 & 0\\ 2 & -1 & 0\\ 1 & -1 & 1 \end{bmatrix}$.

- (i) Find the eigenvalues of *A*.
- (ii) Find a basis for each eigenspace of *A*.
- (iii) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
- 7. True or false?
 - For every square matrix *A* we have det(2A) = 2 det(A).
 - It is impossible to find 5 linearly independent vectors in \mathbb{R}^4 .
 - There is a 3×6 matrix whose nullity is 2.
 - There is a 2 × 2 matrix A with det $(A \lambda I) = \lambda^2 5\lambda 6$.