Math 231 Midterm 2 Review Sheet, 11/6/2025

The second midterm will be on Thursday 11/13 during your usual lecture time. It has four problems (one being a few multiple-choice questions) and is 100 minutes long. You may use a calculator for basic arithmetic although it won't be something you need. *You are not allowed to use any built-in linear algebra packages on a calculator.* Here is a list of the topics that you are advised to study:

- Subspaces of $\mathbb{R}^n$; the concepts of basis and dimension for a subspace; coordinates of a vector relative to a basis.
- Let W be a subspace of $\mathbb{R}^n$. Then the size of any linearly independent set in W is at most the size of any spanning set for W. In particular, if $\dim(W) = k$, every linearly independent set in W contains *at most* k vectors while every spanning set for W contains *at least* k vectors.
- *Basis Test:* Consider n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ and put them as the columns of an $n \times n$ matrix A. Then,
$$\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \text{ is a basis for } \mathbb{R}^n \iff A \text{ is invertible } \iff \det(A) \neq 0.$$
In this case, the coordinate vector of any $\mathbf{b} \in \mathbb{R}^n$ relative to $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is given by the solution of the system with the augmented matrix $[A|\mathbf{b}]$.
- The row space, column space and null space of a matrix; finding bases for these spaces by reducing to a row echelon form; definition of rank and nullity of a matrix; for any $m \times n$ matrix A,
$$0 \leq \operatorname{rank}(A) \leq \min\{m, n\} \qquad \text{and} \qquad \operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$
- Evaluating $\det(A)$ by the cofactor expansion formula along any row or column; determinant of a triangular matrix is the product of its main diagonal entries; $\det(A) = \det(A^T)$; $\det(cA) = c^n \det(A)$ if A is $n \times n$; $\det(AB) = \det(A)\det(B)$; $\det(A^{-1}) = 1/\det(A)$; Cramer's rule.
- Eigenvalues and eigenvectors of a square matrix; eigenvalues of a triangular matrix are its main diagonal entries; finding a basis for each eigenspace; algebraic vs. geometric multiplicity of an eigenvalue; if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of an $n \times n$ matrix A, then
$$\det(A) = \lambda_1 \cdots \lambda_n \quad \text{and} \quad \operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n.$$
- The concept of similar matrices; diagonalizable matrices; an $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors (special case: if there are n distinct eigenvalues); given a diagonalizable matrix A how to find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$; two applications:

- Finding the k-th power of A using the formula $A^k = PD^kP^{-1}$
- Suppose A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then,

$$\mathbf{v} = c_1\,\mathbf{v}_1 + \cdots + c_n\,\mathbf{v}_n \implies A^k\mathbf{v} = c_1\,\lambda_1^k\,\mathbf{v}_1 + \cdots + c_n\,\lambda_n^k\,\mathbf{v}_n.$$

- *Useful characterizations of invertibility*: The following conditions on an $n \times n$ matrix A are equivalent:
  - A is invertible
  - The equation $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$
  - The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$
  - The RREF of A is the identity matrix I
  - $\det(A) \neq 0$
  - $\text{rank}(A) = n$ and $\text{nullity}(A) = 0$
  - The rows (or columns) of A are linearly independent
  - The rows (or columns) of A form a basis for $\mathbb{R}^n$
  - The eigenvalues of A are non-zero

Practice Problems.

1. Verify that $\mathcal{B} = \left\{ \begin{bmatrix} k \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ k \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$ no matter what k is. Then set $k = 1$ and find the coordinate vector of $\begin{bmatrix} x \\ y \end{bmatrix}$ relative to $\mathcal{B}$.

2. Consider the matrix $A = \begin{bmatrix} 3 & -1 & 2 & 0 \\ 1 & 0 & 2 & -1 \\ 5 & -1 & 6 & -2 \end{bmatrix}$.

   (i) Find bases for $\text{row}(A)$ and $\text{col}(A)$ and $\text{null}(A)$.
   (ii) What are $\text{rank}(A)$ and $\text{nullity}(A)$?

3. For what values of x is the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 6 \\ x & 3 & 2 \end{bmatrix}$ singular?

4. Let A, B be $n \times n$ matrices such that $B^{-1}AB = A^2$. What can you say about $\det(A)$?

5. Solve the following system using Cramer's rule:

$$\begin{cases} 2x + 4y + 6z = 18 \\ 4x + 5y + 6z = 24 \\ 3x + y - 2z = 4 \end{cases}$$

6. Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}$.

(i) Find the eigenvalues of A.
(ii) Find a basis for each eigenspace of A.
(iii) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

7. True or false?

- It is impossible to find 5 linearly independent vectors in $\mathbb{R}^4$.
- There is a 3×6 matrix whose nullity is 2.
- There is a 2×2 matrix A with $\det(A - \lambda I) = \lambda^2 - 5\lambda - 6$.
- If A is a 4×4 matrix with $\det(A - \lambda I) = (\lambda - 1)(\lambda + 2)^3$, then $\operatorname{tr}(A) = -1$.