Math 231 Midterm 1 Review Sheet, 3/2/2025

The first midterm will be on Thursday 3/13 during your usual lecture time. It has four problems (one being a few multiple-choice questions) and is 90 minutes long. You are allowed to use a basic calculator but it won't be something you need. Here is a list of the topics that you are advised to study:

- Vectors in $\mathbb{R}^n$ and their components; vector addition and scalar multiplication; dot product and its basic properties; norm and distance; the triangle and Cauchy-Schwarz inequalities; the angle between two vectors; orthogonal vectors; orthogonal projection of one vector onto another.
- Solving systems of linear equations by elimination, i.e., by reducing the associated augmented matrix to a row echelon form; leading and free variables; every consistent (in particular, homogeneous) system with fewer equations than unknowns has infinitely many solutions.
- Linear combinations of vectors; the span of a set of vectors; geometric view of the span in $\mathbb{R}^2$ and $\mathbb{R}^3$; linear dependence versus independence.
- Place the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ as the *columns* of an $n \times k$ matrix A. Then

  $$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{b} \iff x_1, \ldots, x_k \text{ is a solution of } [A|\mathbf{b}].$$

  Thus, we have the following two tests:

  Span Test:

  $$\mathbf{b} \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \iff [A|\mathbf{b}] \text{ has a solution.}$$

  Dependence Test:

  $$\mathbf{v}_1, \ldots, \mathbf{v}_k \text{ dependent} \iff [A|\mathbf{0}] \text{ has infinitely many solutions.}$$

  As a result, every collection of more than n vectors in $\mathbb{R}^n$ must be dependent.
- Addition and multiplication of matrices; the transpose of a matrix; symmetric matrices.
- The inverse of a matrix; if A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$; the formula for the inverse of a 2×2 matrix:

  $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- Elementary matrices; if an elementary row operation applied to I_n gives the elementary matrix E, then the same operation applied to any $n \times n$ matrix A gives the matrix EA.
- Finding the inverse of a matrix by transforming $[A|I]$ to $[I|A^{-1}]$ using elementary row operations.

Practice Problems.

1. Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$$

in $\mathbb{R}^4$. Find the following quantities:

(i) $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $\|\mathbf{u}+\mathbf{v}\|$
(ii) The unit vector in the direction of $\mathbf{u} - \mathbf{v}$.
(iii) The angle between $\mathbf{u}$ and $\mathbf{v}$.
(iv) The scalar t such that $\mathbf{u}$ is orthogonal to $\mathbf{u} + t\mathbf{v}$.
(v) $\mathbf{proj_u}(\mathbf{v})$, the orthogonal projection of $\mathbf{v}$ onto $\mathbf{u}$.

Verify that the triangle and Cauchy-Schwarz inequalities hold for $\mathbf{u}$ and $\mathbf{v}$.

2. Find the general solution of the linear system

$$\begin{cases} x + 3y + 2w = 2 \\ 2x + y + 5z + 4w = -16 \\ 2x + 3y + 3z = 4 \\ 3x + 11y - 2z + 11w = -1 \end{cases}$$

3. Let $A = \begin{bmatrix} 1 & -x \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x \\ 5 & 2 \end{bmatrix}$. Find x such that the product $A^{-1}B^T$ is a symmetric matrix.

4. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 5 \\ 0 & 3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 & 11 \\ 2 & 1 & 5 \\ 1 & -1 & 0 \end{bmatrix}.$$

(i) Explain how B can be obtained by applying two elementary row operations on A.
(ii) Find elementary matrices E_1 and E_2 such that $B = E_2 E_1 A$.

5. Consider the linear system

$$\begin{cases} 2x + 4y + z & = 1 \\ x + 2y & = 2 \\ 3x + 5y + 4z & = 0 \end{cases}$$

(i) Write this system as $A\mathbf{x} = \mathbf{b}$ by introducing the matrices A, $\mathbf{x}$ and $\mathbf{b}$.

(ii) Compute the inverse A^{-1} by applying elementary row operations and use it to find the solution $\mathbf{x} = A^{-1}\mathbf{b}$.

6. True or false?

- Every system of 3 linear equations in 4 unknowns has infinitely many solutions.
- A 9×7 matrix in row echelon form can have up to 9 leading 1's.
- There are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbb{R}^3$ such that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \mathbb{R}^3$.
- For any $m \times n$ matrix A, the product AA^T is defined and symmetric.