

Math 231 Midterm 1 Review Sheet, 9/30/2025

The first midterm will be on Thursday 10/9 during your usual lecture time. It has four problems (one being a few multiple-choice questions) and is 90 minutes long. You are allowed to use a basic calculator but it won't be something you need. Here is a list of the topics that you are advised to study:

- Vectors in \mathbb{R}^n and their components; vector addition and scalar multiplication; dot product and its basic properties; norm and distance; the triangle and Cauchy-Schwarz inequalities; the angle between two vectors; orthogonal vectors; orthogonal projection of one vector onto another.
- Solving systems of linear equations by elimination, i.e., by reducing the associated augmented matrix to a row echelon form; leading and free variables; every consistent (in particular, homogeneous) system with fewer equations than unknowns has infinitely many solutions.
- Linear combinations of vectors; the span of a set of vectors; geometric view of the span in \mathbb{R}^2 and \mathbb{R}^3 ; linear dependence versus independence.
- Place the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in \mathbb{R}^n as the *columns* of an $n \times k$ matrix A . Then

$$x_1 \mathbf{v}_1 + \dots + x_k \mathbf{v}_k = \mathbf{b} \iff x_1, \dots, x_k \text{ is a solution of } [A|\mathbf{b}].$$

Thus, we have the following two tests:

Span Test:

$$\mathbf{b} \in \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \iff [A|\mathbf{b}] \text{ has a solution.}$$

Dependence Test:

$$\mathbf{v}_1, \dots, \mathbf{v}_k \text{ dependent} \iff [A|\mathbf{0}] \text{ has infinitely many solutions.}$$

As a result, every collection of more than n vectors in \mathbb{R}^n must be dependent.

- Addition and multiplication of matrices; the transpose of a matrix; symmetric matrices.
- The inverse of a matrix; if A is invertible, the system $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$; the formula for the inverse of a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- Elementary matrices; if an elementary row operation applied to I_n gives the elementary matrix E , then the same operation applied to any $n \times n$ matrix A gives the matrix EA .
- Finding the inverse of a matrix by transforming $[A|I]$ to $[I|A^{-1}]$ using elementary row operations.

Practice Problems.

1. Consider the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 5 \end{bmatrix}$$

in \mathbb{R}^4 . Find the following quantities:

- $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $\|\mathbf{u} + \mathbf{v}\|$
- The unit vector in the direction of $\mathbf{u} - \mathbf{v}$.
- The angle between \mathbf{u} and \mathbf{v} .
- The scalar t such that \mathbf{u} is orthogonal to $\mathbf{u} + t\mathbf{v}$.
- $\text{proj}_{\mathbf{u}}(\mathbf{v})$, the orthogonal projection of \mathbf{v} onto \mathbf{u} .

Verify that the triangle and Cauchy-Schwarz inequalities hold for \mathbf{u} and \mathbf{v} .

2. Find the general solution of the linear system

$$\begin{cases} x + 3y + 2w = 2 \\ 2x + y + 5z + 4w = -16 \\ 2x + 3y + 3z = 4 \\ 3x + 11y - 2z + 11w = -1 \end{cases}$$

3. Let $A = \begin{bmatrix} 1 & -x \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x \\ 5 & 2 \end{bmatrix}$. Find x such that the product $A^{-1}B^T$ is a symmetric matrix.

4. Let

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & 0 & 5 \\ 0 & 3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 9 \\ 1 & 0 & 5 \\ 1 & -2 & 4 \end{bmatrix}.$$

- Explain how B can be obtained from A by applying two elementary row operations.
- Find elementary matrices E_1 and E_2 such that $B = E_2E_1A$.

5. Consider the linear system

$$\begin{cases} 2x + 3y &= 4 \\ x - 2y - z &= 0 \\ 2x &- z = 1 \end{cases}$$

- (i) Write this system as $A\mathbf{x} = \mathbf{b}$ by introducing the matrices A , \mathbf{x} and \mathbf{b} .
- (ii) Compute the inverse A^{-1} by applying elementary row operations and use it to find the solution $\mathbf{x} = A^{-1}\mathbf{b}$.

6. True or false?

- Every system of 2 linear equations in 3 unknowns has infinitely many solutions.
- A 5×7 matrix in row echelon form can have up to 7 leading 1's.
- There are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in \mathbb{R}^3 such that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \mathbb{R}^3$.
- For any $m \times n$ matrix A , the product AA^T is defined and symmetric.