Math 152 Final Exam Review Sheet

5/9/2025

The final exam will be on Friday 5/16 from 8:00 to 10:30AM in Kiely 170. It will be cumulative, with a slight bias toward the material covered in the second half of the semester. They roughly correspond to methods of integration, applications of the integral, differential equations, and infinite series. Please bring a standard calculator comparable to TI-84.

Below is a list of the topics that you are advised to study. I recommend that you review your lecture notes, the practice problems on our 3 review sheets, and the midterm 1 and 2 problems and solutions. If you have time for extra practice, you can check the homework problems and solutions available on WebAssign.

- Inverse functions, derivative of the inverse function,

$$\text{if} \quad f(a) = b, \quad \text{then} \quad f^{-1}(b) = a \quad \text{and} \quad (f^{-1})'(b) = \frac{1}{f'(a)}$$

- The natural logarithm $\ln x$ and its basic properties, the derivative formula

$$(\ln x)' = \frac{1}{x} \qquad \text{and more generally} \qquad (\ln u)' = \frac{u'}{u}$$

- The exponential function e^x and its basic properties, the derivative formula

$$(e^x)' = e^x \qquad \text{and more generally} \qquad (e^u)' = e^u \cdot u'$$

- Logarithmic differentiation
- Exponential growth and decay
- The inverse trigonometric functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ and their basic properties, the derivative formulas

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}, \quad (\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}}, \quad (\tan^{-1} x)' = \frac{1}{1+x^2}$$

- L'Hospital's rule, its applications in finding limits of indeterminate forms
- The fundamental theorem of calculus, basic integration formulas (for example, review the formulas 1-8, 10, 12, 16, 17 in the table of integrals on reference page 6 in the back of the book)
- Integration by substitution

- Integration by parts
- Integration by partial fractions
- Improper integrals
- Application of integration in finding areas and volumes: Suppose we have two functions f, g such that $0 \leq g(x) \leq f(x)$ for $a \leq x \leq b$. Let R be the region bounded from above by the curve $y = f(x)$, from below by the curve $y = g(x)$, on the left by the line $x = a$ and on the right by the line $x = b$. Then

  □ The area of R is
  $$A = \int_a^b (f(x) - g(x))\, dx.$$
  □ The solid obtained by revolving R about the x-axis has volume
  $$V = \pi \int_a^b (f(x)^2 - g(x)^2)\, dx.$$
  □ Assuming $0 \leq a < b$, the solid obtained by revolving R about the y-axis has volume
  $$V = 2\pi \int_a^b x\,(f(x) - g(x))\, dx.$$
- In general, the volume of a solid (not necessarily of revolution) is the integral of its cross-sectional area function:
  $$V = \int_a^b A(x)\, dx.$$
- The arc length formula: If f is continuously differentiable, the length of the curve $y = f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ is
  $$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$
- Solving first order differential equations by separating variables.
- Sequences and their limits, the squeeze theorem, increasing and decreasing sequences, bounded monotonic sequences are convergent.
- Two useful limits:
  $$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a \qquad\qquad \lim_{n\to\infty} \sqrt[n]{n} = 1.$$
- The geometric series:
  $$\sum_{n=0}^{\infty} r^n \begin{cases} \text{converges to } \frac{1}{1-r} & \text{if } |r| < 1 \\ \text{diverges} & \text{if } |r| \geq 1. \end{cases}$$

- The p-series:
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$
In particular, the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges.
- Basic divergence test: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- The integral, comparison, and limit comparison tests for series with positive terms.
- The ratio and root tests for series with positive terms.
- The Leibniz test for alternating series, absolute and conditional convergence, an absolutely convergent series is convergent.
- Power series, finding the radius and interval of convergence using the ratio test.
- Manipulating power series, within the interval of convergence a power series can be differentiated and integrated term-by-term.
- Taylor's formula:
$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$
where the remainder R_n has the form
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } a \text{ and } x.$$
If $\lim_{n\to\infty} R_n(x) = 0$ for all x in the interval $(a-R, a+R)$, then f is equal to its Taylor series:
$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots \end{aligned}$$
inside the interval $(a-R, a+R)$.
- Taylor series of a few basic functions about $a = 0$ (also known as their Maclaurin series). The most important examples are given below, but other examples can be constructed using these power series, as we discussed in class:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad -\infty < x < +\infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad -\infty < x < +\infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad -\infty < x < +\infty$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad -1 < x < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots \qquad -1 < x < 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad -1 < x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad -1 < x < 1$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad -1 < x < 1.$$

- Applications of Taylor series in approximating values of functions or integrals.

Practice Problems

1. In each case, find the derivative $y' = dy/dx$:

- $y = \sin^{-1}(x + e^x)$
- $y = \sqrt{\ln(\cos x)}$
- $y = x^{\tan x}$ [Hint: Logarithmic differentiation]

2. Find $\lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2}$. [Hint: L'Hospital]

3. Evaluate the following integrals:

- $\int \frac{dx}{x(\ln x)^4}$ [Hint: u-substitution]
- $\int \sqrt{2 - x^2}\, dx$ [Hint: Trigonometric substitution]
- $\int \frac{3x - 1}{x^2 + 3x - 10}\, dx$ [Hint: Partial fractions]

4. Let R be the region in the plane bounded by the curve $y = e^x$ and the lines $y = x + 1$ and $x = 2$.

(i) Sketch R and find its area.
(ii) Find the volume of the solid obtained by rotating R about the x-axis.
(iii) Find the volume of the solid obtained by rotating R about the y-axis.

5. Set up an integral for the arc length of $y = \tan x$ for $0 \leq x \leq \pi/3$. Then use your calculator to estimate this length to four decimal places.

6. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{1+x}{xy^2}$$

where $x > 0$. Then find the solution that satisfies the condition $y(1) = 3$.

7. Determine the convergence or divergence of the following series:

- $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{7n+3}$
- $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{11^n\, n!}{4^n\, n^n}$

8. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{5^n}{n^2} (x+3)^n.$$

9. Find the first four terms in the Maclaurin series of the following functions:

- $x^2 \cos(5x)$
- $\displaystyle\int \frac{dx}{2+x^3}$

10. Find the Maclaurin series of e^{-x^2} and use it to write the integral

$$\int_0^{0.5} e^{-x^2}\, dx$$

as an alternating series. Use this series to estimate the value of the above integral with an error of less than 0.00001.